SUPERGEOMETRY AND QUANTUM FIELD THEORY, OR: WHAT IS A CLASSICAL CONFIGURATION?

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ABSTRACT. We discuss of the conceptual difficulties connected with the anticommutativity of classical fermion fields, and we argue that the "space" of all classical configurations of a model with such fields should be described as an infinite-dimensional supermanifold M. We discuss the two main approaches to supermanifolds, and we examine the reasons why many physicists tend to prefer the Rogers approach although the Berezin-Kostant-Leites approach is the more fundamental one. We develop the infinite-dimensional variant of the latter, and we show that the superfunctionals considered in [44] are nothing but superfunctions on M. We propose a programme for future mathematical work, which applies to any classical field model with fermion fields. A part of this programme will be implemented in the successor paper [45].

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Introduction

Although this paper logically continues [44], it can be read independently.

We begin with a discussion of the conceptual difficulties connected with the anticommutativity of classical (=unquantized) fermion fields, and we argue that only a supergeometric approach allows a convincing description of these fields; indeed, the "space" of all classical configurations of a model with such fields should be described as an infinite-dimensional supermanifold.

Before explicating this, we discuss the two main approaches to supermanifolds, starting with the approach of Berezin, Leites, Kostant et al, together with the hermitian modification proposed in [39]. We then review the second, alternative approach proposed by B. deWitt and A. Rogers and followed by quite a few authors and its connection to the first one as well as the "interpolating" approach of Molotkov ([29]), and we examine the reasons why many physicists tend to prefer it although the Berezin-Kostant-Leites approach is the more fundamental one.

In particular, we show that the deWitt-Rogers supermanifolds suffer from the same conceptual shortcomings as the naive configuration notion in classical field theory with anticommuting fields.

The deWitt-Rogers approach is intimately connected with an "interim solution" of the conceptual difficulties mentioned: one considers "B-valued configurations", i. e. instead of the usual real or complex numbers, one uses a Grassmann algebra B (or some relative of it) as target for the classical fields, providing in this way the apparently needed "anticommuting values". However, this approach is suspicious for the very same reasons as the deWitt-Rogers supermanifold approach; in particular, in view of the arbitrariness in the choice of B. Therefore we go one step beyond and re-interpret B-valued configurations as *families of configurations* parametrized by supermanifolds. Once this is done, it is natural to state the question for a universal family and a corresponding moduli space – an idea which directly leads to the infinite-dimensional configuration supermanifold mentioned.

Hence, we have to develop the infinite-dimensional variant of the Berezin-Kostant-Leites approach, basing on previous work of the author. In fact, we will show that the superfunctionals considered in [44] are nothing but *superfunctions on an infinite-dimensional supermanifold*. If the model is purely bosonic then a superfunctional is an ordinary functional, i. e. a function the domain of definition of which is a function space.

Generalizing the philosophy of the paper [5] from quantum mechanics onto quantum field theory, we propose a programme for future mathematical work, which applies to any classical field model with fermion fields, and which aims at understanding its mathematical structure. A part of this programme will be implemented in the successor paper [45].

1. Supergeometry and its relationship to quantum field theory

1.1. **Geometric models of quantum field theory and classical configurations.** The modern realistic quantum field theoretic models, like quantum electrodynamics, Salam-Weinberg electroweak theory, quantum chromodynamics, and all these unified in the standard model, have achieved remarkable success in predicting experimental results, in one case even with an accuracy up to 11 digits. Even for a mathematician, which turns up his nose about the mathematical and logical status ot these models, this should be strong evidence that these models have something to do with nature, and that they should be taken seriously. Even if it should turn out that they do not produce rigorous quantum theories in Wightman's sense, they have proven to give at least a kind of semiclassical limit plus quite a few quantum corrections of a more fundamental theory (provided the latter exists), and therefore it remains a promising task to clear up the mathematical structure, and to put it into a consistent framework as far as possible. Moreover, there exist phenomenological models in hadron and nuclear physics which are rather unlikely to be implementable as rigorous quantum theories, due to non-renormalizability; nevertheless, they give information, and thus should be taken serious.

The models mentioned start with formulating a *classical* field theory of fields on Minkowski space $\mathbb{R}^4$, and the hypothetical final theory is thought to arise by quantization of the classical theory in a similar sense as quantum mechanics arises by quantization of point mechanics. Of course, the concept of quantization is problematic even in the latter situation; but Kostant-Souriau geometric quantization gives at least one mathematically well-defined procedure.

In the field theoretic situation, where we have infinitely many degrees of freedom, geometric quantization in its usual form does no longer apply, and there arises a gap between mathematics and physics: The models which have been constructed rigorously (like the $P(\Phi)_d$ models, the Thirring model, the Gross-Neveu model (cf. [31], [19] and references therein)), are only toy models in small space-time dimensions, while the realistic models mentioned above are treated up to now only heuristically (this statement is not affected by the fact that some special features have been successfully treated by rigourous mathematical methods, like e. g. the instanton solutions of euclidian pure Yang-Mills theory).

A popular device to tackle the field quantization problem heuristically is Feynman's path integral. Inspite of the fact that only some aspects are mathematically understood (cf. e. g. Osterwalder-Schrader axiomatics a la Glimm-Jaffe [17]), and that only the bosonic part can be understood as integration over a measure space, it has been very successful in deriving computation rules by formal manipulation like variable changing, integration by parts, standard Gaussian integrals.

Now the path integral is supposed to live on the space M of classical configurations Ξ of the model: under some restrictions onto the Lagrangian density $\mathcal{L}(\Xi)$, "each one" of them enters formally with the weight $\exp(\mathrm{i}\int d^4x\, \mathcal{L}[\Xi](x)/\hbar)$ (or, in the euclidian variant, minus instead of the imaginary unit).

Now, if fermion fields enter the game then a complication arises (and this applies in the standard models both to the matter fields and to the ghost fields in the Faddeev-Popov scheme): Any classical fermion field Ψ_α has to be treated as anticommuting,

$$[\Psi_\alpha(x), \Psi_\beta(y)]_+ = 0 \tag{1.1.1}$$

($x, y \in \mathbb{R}^4$ are space-time points; physicists use to say that $\Psi_\alpha(x)$ is an "a-number"), and this makes the concept of a "classical configuration" for fermion fields problematic. ("Fermion" refers always to the statistics, not to the spin. The difference matters for ghost fields, which do not obey the Pauli Theorem.)

In [3], Berezin developed a recipe to compute Gaussian functional integrals over fermions in a heuristic way (actually, his recipe is mathematically precise but applies only to rather "tame" Gaussian kernels; however, the practical application is heuristic because the kernels are far from being "tame"). Thus, in the standard models, one can often circumvent the problem mentioned above by "throwing out" the fermions by Gaussian integration, so that only bosonic configurations remain. (However, the

Faddeev-Popov scheme for the quantization of gauge theories does the opposite: after some formal manipulations with the bosonic functional integral, a determinant factor appears which is represented by a fermionic Gaussian integral. The rest is Feynman perturbation theory.)

On the other hand, the problem of understanding the notion of a classical configuration becomes urgent in supersymmetric models, because – at least in component field formulation – supersymmetry simply does not work with the naive notion of configuration with commuting fields, and the computations of physicists would not make sense. Only anticommuting fields behave formally correct.

Also, in the mathematical analysis of the BRST approach to the quantization of constrained systems (cf. e. g. [24]), the fermionic, anticommuting nature of the ghost degrees of freedom (ghost fields in the context of Yang-Mills theory) has to be taken into account.

Contrary to naive expectations, the law (1.1.1) is not automatically implemented by considering superfield models on supermanifolds: The components $f_\mu(x)$ of a superfunction

$$f(x,\xi)=\sum f_\mu(x)\xi^\mu$$

are still $\mathbb{R}$- (or $\mathbb{C}$-) valued and therefore commuting quantities while e. g. for the chiral superfield (cf. e. g. [51, Ch. V], therein called "scalar superfield")

$$\Phi(y,\theta)=A(y)+2\theta\psi(y)+\theta\theta F(y), \tag{1.1.2}$$

the Weyl spinor $\psi(x)$, which describes a left-handed fermion, has to satisfy the law (1.1.1) – at least if non-linear expressions in ψ have to be considered.

Thus, in the framework of superfield models, it still remains as obscure as in component field models how to implement (1.1.1) mathematically. Nevertheless, we will see that it is just supergeometry which is the key to an appropriate solution of this problem. But before establishing the link, we review the history and the main approaches to supergeometry.

Unfortunately, this problem has been mostly neglected in the existing literature both on super differential geometry and on mathematical modelling of field theories with fermions. In the worst case, fermions were modelled by ordinary sections in spinor bundles, like e. g. in [12]. (Of course, this is okay as long as all things one does are linear in the fermion fields. For instance, this applies to the vast literature on the Dirac equation in a gauge field background. On the other hand, if e. g. the first Yang-Mills equation with spinor current source or supersymmetry transformations have to be considered then this approach becomes certainly inconsistent).

1.2. **Remarks on the history of supergeometry.** "Supermathematics", and, in particular, super differential geometry, has its roots in the fact (which is familiar and unchallenged among physicists, although not seldom ignored by mathematicians) that in the framework of modern quantum field theory, which uses geometrically formulated field theories, fermions are described on a classical level by anticommuting fields. Indeed, practical experience in heuristic computations showed that if keeping track of the signs one can handle anticommuting fields formally "in the same way" as commuting (bosonic) ones. The most striking example for this is the Feynman path integral, or its relative, the functional integral, for fermion fields; cf. [3]. This experience led F. A. Berezin to the conclusion that "there exists a non-trivial analogue of analysis in which anticommuting variables appear on equal footing with the usual variables" (quotation from [4]), and he became the pioneer in constructing this theory. The book [4] which summarizes Berezin's work in this field, reflects supergeometry "in statu nascendi".

The papers [23], [25], and the book [26] give systematic, methodologically closed presentations of the foundations of supergemetry; cf. [25] also for more information on the history of the subject.

In the meantime, supergeometry turned to more special questions, in particular, to the investigation of super-analogues of important classical structures. As examples from the vast literature, let us mention Serre and Mekhbout duality on complex supermanifolds ([30]), Lie superalgebras and their

representations (see e. g. [21], [35]), integration theory on supermanifolds ([6], [7], [33]), deformations of complex supermanifolds ([49]), integrability of CR structures ([37], [38]), the investigation of twistor geometry and Penrose transform in the super context ([27] and references therein). (Of course, the quotations are by no means complete.) Manin's book [27] seems to be one of the most far-reaching efforts to apply supergeometry to classical field models in a mathematical rigorous way.

Also, one should mention the "Manin programme" [28] which calls for treating the "odd dimensions" of super on equal footing with the ordinary "even dimensions" as well as with the "arithmetic" or "discrete dimensions" of number theory, and for considering all three types of dimensions in their dialectic relations to each other.

Now if speaking on the history of supergeometry one should also mention the alternative approach of B. deWitt and A. Rogers to supermanifolds, which was followed (and often technically modified) by several authors (cf. e.g. [20], [50], [14]), and the theory of Molotkov [29] which in some sense unifies both approaches in an elegant way and allows also infinite-dimensional supermanifolds. We will comment this below.

1.3. **A short account of the Berezin-Leites-Kostant approach.** Roughly speaking, a *finite-dimensional supermanifold* (*smf*) X is a "geometrical object" on which there exist locally, together with the usual even coordinates $x_1, \ldots, x_m$, odd, anticommuting coordinates $\xi_1, \ldots, \xi_n$.

In order to implement this mathematically, one defines a *(smooth) superdomain* U of dimension $m|n$ as a ringed space $U = (\mathrm{space}(U), \mathcal{O}_\infty)$ where $\mathrm{space}(U) \subseteq \mathbb{R}^m$ is an open subset, and the structure sheaf is given by

$$\mathcal{O}_\infty(\cdot) := C^\infty(\cdot) \otimes_{\mathbb{R}} \mathbb{R}[\xi_1, \ldots, \xi_n] \tag{1.3.1}$$

where $\xi_1, \ldots, \xi_n$ is a sequence of Grassmann variables. Thus, $\mathcal{O}_\infty(V)$ for open $V \subseteq U$ is the algebra of all formal sums

$$f(x|\xi) = \sum f_{\mu_1 \cdots \mu_n}(x) \xi^{\mu_1} \cdots \xi^{\mu_n} = \sum f_\mu(x) \xi^\mu \tag{1.3.2}$$

where the sum runs over all 2^n tuples $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_2^n$, and the $f_\mu(x)$ are smooth functions $V \to \mathbb{R}$. One interprets (1.3.2) as the Taylor expansion of $f(x|\xi)$ w. r. to the Grassmann variables; due to their anticommmutativity, it terminates after 2^n terms.

Here and in the following, we use the subscript ∞ at the structure sheaf in order to emphasize that we are considering the smooth variant of the theory; the symbol $\mathcal{O}$ without subscript is reserved to complex- and real-analytic structure sheaves, as already considered in [44, 3.5]. Note that in the finite-dimensional situation, the definitions of the complex- and real-analytic variants of (1.3.1) are straightforward. The "globality theorems" like [23, Prps. 2.4.1, 2.4.2, etc.], or the quirk of [23] of defining morphisms by homomorphisms of the algebra of global superfunctions, do no longer work in these situations. However, if one instead follows the standard framework of ringed spaces there is no genuine obstackle to a satisfactory calculus.

Although the use of a *real* Grassmann algebra in (1.3.1) looks quite natural, it is in fact highly problematic; we will discuss this in 1.5 below.

Now one defines a *supermanifold (smf)* as a ringed space $X = (\mathrm{space}(X), \mathcal{O}_\infty)$ with the underlying space paracompact and Hausdorff which is locally isomorphic to a superdomain. (Kostants definition is easily seen to be equivalent to this one.) Morphisms of smfs are just morphisms of ringed spaces. We will not mention here the various differential-geometric ramifications; in section 2, we will develop the infinite-dimensional variant of some of them. We only note that every smf determines an *underlying C^∞ manifold*, and that, conversely, every C^∞ manifold can be viewed as smf.

For later use we note also that to every finite-dimensional $\mathbb{Z}_2$-graded vector space $V = V_{\mathbf{0}} \oplus V_{\mathbf{1}}$ there belongs a *linear superspace* (often called also "affine superspace", in analogy with the habits of algebraic geometry) $\mathrm{L}(V)$ which has the even part $V_{\mathbf{0}}$ as underlying space and $C^\infty(\cdot) \otimes \Lambda V_{\mathbf{1}}^*$ as

structure sheaf. Here $\Lambda V_{\mathbf{1}}^*$ is the exterior algebra over the dual of the odd part; due to the dualization, one has a natural embedding of the whole dual into the global superfunctions,

$$V^* \subseteq \mathcal{O}_\infty(\mathrm{L}(V)), \tag{1.3.3}$$

and V^* is just the space of *linear superfunctions*. This procedure is just a superization of the fairly trivial fact that every finite-dimensional vector space can be viewed as smooth manifold.

It is of outstanding importance that a morphism from an arbitrary smf X to a linear smf $\mathrm{L}(V)$ is known by knowing the coordinate pullbacks: Fix a basis $u_1, \ldots, u_k \in V_{\mathbf{0}}^*, \quad \xi_1, \ldots, \xi_l \in V_{\mathbf{1}}^*$ of the dual V^*; because of (1.3.3), these elements become superfunctions $u_1, \ldots, u_k, \xi_1, \ldots, \xi_l \in \mathcal{O}_\infty(\mathrm{L}(V))$. It is reasonable to call this tuple a *coordinate system* on $\mathrm{L}(V)$, and we have (cf. [25, Thm. 2.1.7], [36, Thm. 3.1], [39, Cor. 3.3.2]):

Theorem 1.3.1. *Given superfunctions $u'_1, \ldots, u'_k \in \mathcal{O}_\infty(X)_{\mathbf{0}}, \quad \xi'_1, \ldots, \xi'_l \in \mathcal{O}_\infty(X)_{\mathbf{1}}$ there exists a unique morphism $\mu : X \to \mathrm{L}(V)$ such that $\mu^*(u_i) = u'_i, \quad \mu^*(\xi_j) = \xi'_j$ for all i, j.* □

This Theorem makes the morphisms between the local models controllable. One can give it a more invariant form:

Corollary 1.3.2. *Given an smf X, a finite-dimensional $\mathbb{Z}_2$-graded vector space V, and an element $\hat{\mu} \in (\mathcal{O}_\infty(X) \otimes V)_{\mathbf{0}}$ there exists a unique morphism $\mu : X \to \mathrm{L}(V)$ such that for any $v \in V^*$ we have $\langle v, \hat{\mu} \rangle = \mu^*(v)$ within $\mathcal{O}_\infty(X)$.* □

In this form, it will also generalize to the infinite-dimensional situation (cf. Thm. 2.8.1 below); however, the real-analytic analogon $\mathcal{O}(\cdot) \otimes V$ of the sheaf $\mathcal{O}_\infty(\cdot) \otimes V$ will have to be replaced by a bigger sheaf $\mathcal{O}^V(\cdot)$ of "V-valued superfunctions".

Remark . Several authors, like e. g. [29], consider generalized frameworks based on a $\mathbb{Z}_2^N$ grading, so that the parity rule as well as the first sign rule have to be applied to each degree $|\cdot|_j$ separately. Thus, a $\mathbb{Z}_2^N$-graded algebra $R = \bigoplus_{(i_1,\ldots,i_N)\in\mathbb{Z}_2^N} R_{(i_1,\ldots,i_N)}$ is $\mathbb{Z}_2^N$-*commutative* iff

$$ba = (-1)^{|a|_1|b|_1 + \ldots |a|_N|b|_N} ab$$

for homogeneous $a, b \in R$. Some people claim that the use of such a structure is appropriate in order to separate physical fermion field d. o. f.'s from ghost field d. o. f.'s.

However, as observed e. g. in [18, Ex. 6.15], one can work without such a generalization, thus saving at least some notations: One may construct from R a $\mathbb{Z}_2$-graded algebra $R_{\mathbf{s}}$ by taking the total degree,

$$(R_{\mathbf{s}})_{\mathbf{k}} := \bigoplus_{(i_1,\ldots,i_N)\in\mathbb{Z}_2^N:\ i_1+\cdots+i_N \equiv \mathbf{k} \mod 2} R_{(i_1,\ldots,i_N)}$$

for $\mathbf{k} = \mathbf{0}, \mathbf{1}$; thus $|a|_{\mathbf{s}} = \sum_i |a|_i$. Also, one modifies the multiplication law by

$$a \cdot_{\mathbf{s}} b := (-1)^{\sum_{i>j} |a|_i\, |b|_j} ab$$

for homogeneous elements. This new product is again associative; if R was $\mathbb{Z}_2^N$-commutative then $R_{\mathbf{s}}$ is $\mathbb{Z}_2$-commutative. Analogous redefinitions can be made for modules and other algebraic structures, so that there is no real necessity to consider such generalized gradings.

In particular, instead of constructing the exterior differential on an smf as even operator (cf. e. g. [23]), which leads to a $\mathbb{Z}_2 \times \mathbb{Z}_+$-graded commutative algebra of differential forms, it is more appropriate to use an odd exterior differential. This leads to an algebra of differential forms which is still $\mathbb{Z}_2 \times \mathbb{Z}_+$-graded but where only the $\mathbb{Z}_2$degree produces sign factors. Such an approach does not only simplify computations but it opens the way for a natural supergeometric interpretation of exterior, interior, and Lie derivatives as vector fields on a bigger smf, and it leads straightforwardly to the Bernstein-Leites pseudodifferential forms. Cf. [6], [36], [42].

1.4. **Supergeometry and physicists.** Supergeometry has both its sources and its justification mainly in the classical field models used in quantum field theory. Nowadays it is most used for the formulation and analysis of supersymmetric field models in terms of superfields which live on a supermanifold (cf. e. g. [51], [27]). However, most of the physicists who work in this field either do not make use of the mathematical theory at all, or they use only fragments of it (mainly the calculus of differential forms and connections, and the Berezin integral over volume forms).

The first and most obvious reason for this is that one can do formulation and perturbative analysis of superfield theories without knowing anything of ringed spaces (one has to know how to handle superfunctions $f(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha})$ which depend on commuting coordinates x^μ and anticommuting coordinates θ^α, $\overline{\theta}^{\dot\alpha}$; for most purposes it is inessential how this is mathematically implemented).

The second reason is fairly obvious, too: Despite the invasion of manifolds, fibre bundles, cohomology, Chern classes etc. into mathematical physics – the work with e. g. morphisms (instead of "genuine maps" of sets) or group objects (instead of "genuine groups") is not easy even for the mathematically well-educated physicist. However, the difficulties are of purely psychological nature; the algebraic geometer has to deal with quite similar (and sometimes still far more abstract) structures since Grothendieck's revolution in this area (one should mention here that also C^∞ supergeometry has learned a lot from algebraic geometry).

But there are also two further, less obvious reasons. One of them is a certain conceptual shortcoming of Berezin's theory concerning the treatment of complex conjugation, which made it almost inapplicable to physical models. In [39], the present author made a proposal to modify this theory in order to adapt it to the needs of quantum physics; cf. 1.5 below for a short account.

Finally, the fourth, and perhaps most severe reason is the problem of modelling classical fermion fields mentioned above. In particular, the Berezin-Kostant-Leites approach does not provide anticommuting constants, which seem to be indispensable (but are, as we will see in 1.11, in fact not).

Although supergeometry is usually considered to be merely a tool for the formulation of supersymmetric models in terms of superfields on supermanifolds, its conceptual importance is much wider: while the x_i describe usual, "bosonic" degrees of freedom the ξ_j describe new, "fermionic" degrees of freedom which are not encountered in classical mechanics. In fact, *supergeometry is the adequate tool to describe the classical limit of fermionic degrees of freedom.*

This point of view is pursued also in the paper [5] of Berezin and Marinov. However, there seems to be a minor conceptual misunderstanding in that paper which we want to comment on: The authors argue that one can describe the algebra of classical observables of a spinning electron in an external field as a Grassmann algebra $R = \mathbb{C}[\xi_1, \xi_2, \xi_3]$ (or, more precisely, as the even, real part of R). Time evolution of observables $f \in R$ is governed by $df/dt = \{H, f\}$ where $\{\cdot, \cdot\}$ is a Poisson bracket making R a Lie superalgebra, and $H \in R_{\mathbf{0}}$ is the Hamiltonian. In supergeometric language, the phase space is the $0|3$-dimensional hermitian linear supermanifold (cf. 1.5) $\mathrm{L}(\mathbb{R}^{0|3})$ equipped with the symplectic structure $\omega = (d\xi_1)^2 + (d\xi_2)^2 + (d\xi_3)^2$ which induces the Poisson bracket in $R = \mathcal{O}_\infty(\mathrm{L}(\mathbb{R}^{0|3}))$.

Now the notion of "phase space trajectory" introduced in [5, 2.1] as a smooth map $\mathbb{R} \to R_{\mathbf{1}}$, $t \mapsto \xi(t)$, leads to confusion. Indeed, $t \mapsto \xi(t)$ describes the time evolution of the observable $\xi(0)$ rather than that of a state (that is, we have a "Heisenberg picture" at the unquantized level).

In accordance with this, for the "phase space distribution" ρ of [5, 2.2], there is no sensible notion of a δ-distribution which would indicate a "pure state". Thus, it should be stressed that in this approach – as well as in our one to be developed below – there do not exist "individual configurations" on the unquantized level.

With this point of view on [5], one may say that the programme presented in section 1.12 below is the logical extension of the philosophy of [5] from classical mechanics onto classical field theory.

For a systematic (but rather formal) treatment of supermanifolds as configuration spaces of mechanical systems, cf. the book [18] and references therein.

In 1.13, we will see how supermanifolds emerge naturally as configuration spaces of lattice theories; the advantage of those is that the finite-dimensional calculus is sufficient.

Now if we consider a *field* (instead of, say, a mass point) then we will have infinitely many degrees of freedom. Roughly speaking, the bosonic and fermionic field strengthes $\Phi_i(x)$, $\Psi_j(x)$ for all space-time points x are just the coordinates of the configuration space.

The appearance of anticommuting functions on the configuration space indicates that the latter should be understood as an *infinite-dimensional supermanifold*. We will elaborate this philosophy from 1.11 on.

However, the use of anticommuting variables is not restricted to the unquantized theory: "Generating functionals" of states and operators in Fock space were invented by F. A. Berezin [3] thirty years ago. While in the purely bosonic case they are analytic functions on the one-particle state space H, their geometric interpretation remained up to now obscure on the fermionic side: they are just elements of an "infinite-dimensional Grassmann algebra" the origin of which was not clear.

Infinite-dimensional supergeometry provides a satisfactory solution of the riddle: the Grassmann algebra mentioned above is nothing but the algebra of superfunctions on the $0|\infty$-dimensional supermanifold $\mathrm{L}(H)$. More generally, supergeometry allows to unify bosonic and fermionic case, instead of considering them separately, as Berezin did. Cf. [43] for a sketch; a detailed presentation will appear elsewhere.

Another natural appearance of supergeometry is in quantized euclidian theory, which is given by its *Schwinger functions*, i. e. the vacuum expectation values of the quantized fields:

$$\begin{aligned} S^{I|J}(X|Y) &= S^{i_1,\dots,i_k|j_1,\dots,j_l}(x_1,\dots,x_k|y_1,\dots,y_l) := \Big\langle \hat\Phi_{i_1}(x_1)\cdots\hat\Phi_{i_k}(x_k)\hat\Psi_{j_1}(y_1)\cdots\hat\Psi_{j_l}(y_l)\Big\rangle_{\mathrm{vac}} \\ &= \frac{1}{N}\int [D\Phi][D\Psi]\Phi_{i_1}(x_1)\cdots\Phi_{i_k}(x_k)\cdot\Psi_{j_1}(y_1)\cdots\Psi_{j_l}(y_l)\exp(\frac{-1}{\hbar}S_{\mathrm{euc}}[\Phi|\Psi]). \end{aligned}$$

Here $S_{\mathrm{euc}}[\Phi|\Psi]$ is the euclidian action which depends on bosonic fields Φ_i and fermionic ones Ψ_j (for notational simplicity, we use real field components).

Usually, one assumes the Schwinger functions to be tempered distributions defined on the whole space, $S^{I|J} \in \mathcal{S}'(\mathbb{R}^{d(k+l)})$, and satisfying the Osterwalder-Schrader axioms (cf. [8, 9.5.B]). The Schwinger functions now are the coefficient functions (cf. [44, 2.3]) of the *Euclidian generating functional*

$$\begin{aligned} Z_{\mathrm{euc}}[\mathrm{J}^\Phi\,|\,\mathrm{J}^\Psi] &= \sum_{k,l\ge 0}\frac{1}{k!l!}\sum_{i_1,\dots,i_k,j_1,\dots,j_l}\int dX\,dY\,S^{I|J}(X|Y)\,\mathrm{J}^\Phi_{i_1}(x_1)\cdots\mathrm{J}^\Phi_{i_k}(x_k)\,\mathrm{J}^\Psi_{j_1}(y_1)\cdots\mathrm{J}^\Psi_{j_l}(y_l) \\ &= \frac{1}{N}\int [D\Phi][D\Psi]\exp\Big(\frac{-1}{\hbar}S_{\mathrm{euc}}[\Phi|\Psi] + \int dx(\sum_i \mathrm{J}^\Phi_i(x)\Phi_i(x) + \sum_j \mathrm{J}^\Psi_j(x)\Psi_j(x))\Big) \end{aligned}$$

where the functional variables $\mathrm{J}^\Phi, \mathrm{J}^\Psi$ are (formal) "external sources"; note that J^Φ, J^Ψ are commuting and anticommuting, respectively.

In the purely bosonic case, $Z_{\mathrm{euc}}[\mathrm{J}^\Phi] : \mathcal{S}(\mathbb{R}^d)\otimes V^* \to \mathbb{C}$ is the characteristic function of a measure on the euclidian configuration space $\mathcal{S}'(\mathbb{R}^d)\otimes V$ of the theory; here V is the field target space as in [44, 2.2]. However, if fermionic fields are present then, due to the antisymmetry of the Schwinger functions in the fermionic sector, $Z_{\mathrm{euc}}[\mathrm{J}^\Phi\,|\,\mathrm{J}^\Psi]$ cannot be interpreted as a map any longer. Instead of this, it is at least a formal power series in the sense of [44, 2.3], and thanks to the usual regularity condition on the Schwinger functions, it becomes a superfunction $Z_{\mathrm{euc}}[\mathrm{J}^\Phi\,|\,\mathrm{J}^\Psi] \in \mathcal{O}(\mathrm{L}(\mathcal{S}(\mathbb{R}^d)\otimes V^*))$. (It is a challenging task to construct some superanalogon of measures, so that $Z_{\mathrm{euc}}[\mathrm{J}^\Phi\,|\,\mathrm{J}^\Psi]$ becomes the characteristic superfunction of a supermeasure on the euclidian configuration supermanifold $\mathrm{L}(\mathcal{S}'(\mathbb{R}^d)\otimes V)$ of the theory.)

Apart from the last remark, analogous remarks apply also to the (time-ordered) Green functions in Minkowski theory. Note, however, that, although their existence is generally assumed, it does not follow from the Wightman axioms; cf. also [8, 13.1.A].

1.5. **Supergeometry and hermitian conjugation.** Here, we give a short account of [39].

Usually, one uses the sign rule in the following form:

Sign Rule. Whenever in a multilinear expression standing on the r.h.s. of an equation two adjacent terms A, B are interchanged (with respect to their position on the r. h. s.) a sign $(-1)^{|A||B|}$ occurs.

This rule has is origin in the commutation rule of Grassmann algebra, and it is well-known whereever these are applied, e. g. in supergeometry as well as in homological algebra, differential geometry (in treating exterior forms), and algebraic topology.

However, the law of operator conjugation in the quantized theory, $(AB)^* = B^*A^*$ yields in the classical limit the rule $\overline{fg} = \overline{g}\,\overline{f}$ for superfunctions f, g on a supermanifold. This seems to contradict the sign rule, and the best way out of the trouble is the following:

One applies the sign rule only to *complex* multilinear expressions. In fact, one avoids the use of merely real-linear terms; that is, all vector spaces the elements of which appear in multilinear terms should be complex. With this first step, the contradiction is resolved, but there remains incertainty. Thus, we have to do more.

Skew-linearity should appear only in the form of explicit hermitian conjugation, and the latter is treated by:

Second Sign Rule. If conjugation is applied to a bilinear expression containing the adjacent terms a, b (i. e. if conjugation is resolved into termwise conjugation), either a, b have to be rearranged backwards, or the expression acquires the sign factor $(-1)^{|a||b|}$. Multilinear terms have to be treated iteratively.

These rules have consequences for the very definition of supermanifolds: In a natural way, one defines a *hermitian ringed space* as a pair $X = (\mathrm{space}(X), \mathcal{O}_\infty)$ where $\mathrm{space}(X)$ is a topological space, the structure sheaf $\mathcal{O}_\infty$ is now a sheaf of *complex* $\mathbb{Z}_2$-graded algebras which are equipped with a *hermitian conjugation*, i. e. an antilinear involution $^- : \mathcal{O}_\infty \to \mathcal{O}_\infty$ which satisfies

$$\overline{fg} = \overline{g}\,\overline{f} \qquad \text{for } f, g \in \mathcal{O}_\infty.$$

Now one defines a *hermitian superdomain* U of dimension $m|n$ as a hermitian ringed space $U = (\mathrm{space}(U), \mathcal{O}_\infty)$ where $\mathrm{space}(U) \subseteq \mathbb{R}^n$ is open again but

$$\mathcal{O}_\infty(\cdot) = C^\infty_{\mathbb{C}}(\cdot) \otimes_{\mathbb{C}} \mathbb{C}[\xi_1, \ldots, \xi_n]$$

where $\xi_1, \ldots, \xi_n$ is again a sequence of Grassmann variables. Thus, $\mathcal{O}_\infty(V)$ for open $V \subseteq U$ is the algebra of all formal sums (1.3.2) where, however, the smooth functions $f_\mu(x)$ are now complex-valued. The hermitian conjugation is defined by the properties

$$\overline{f} = f \quad \text{for } f \in C^\infty_{\mathbb{R}}, \qquad \overline{\xi}_i = \xi_i \quad \text{for all } i.$$

Thus

$$\overline{f(x|\xi)} = \sum (-1)^{|\mu|(|\mu|-1)/2}\, \overline{f_\mu(x)}\, \xi^\mu. \tag{1.5.1}$$

Now one defines a *hermitian supermanifold* as a hermitian ringed space with the underlying space paracompact and Hausdorff which is locally isomorphic to a hermitian superdomain. Note that the definition of the linear superspace belonging to a finite-dimensional $\mathbb{Z}_2$-graded vector space $V = V_{\mathbf{0}} \oplus V_{\mathbf{1}}$ also changes:

$$\mathrm{L}(V) = (V_{\mathbf{0}},\ C^\infty_{\mathbb{C}}(\cdot) \otimes_{\mathbb{C}} \Lambda V^*_{\mathbf{1},\mathbb{C}}).$$

Thm. 1.3.1 slightly changes because the coordinate pullbacks now have to be required real: $u'_1, \ldots, u'_k \in \mathcal{O}_\infty(X)_{\mathbf{0},\mathbb{R}}$, $\xi'_1, \ldots, \xi'_l \in \mathcal{O}_\infty(X)_{\mathbf{1},\mathbb{R}}$. Also, in Cor. 1.3.2, the element $\hat{\mu}$ has to be required real: $\hat{\mu} \in (\mathcal{O}_\infty(X) \otimes V)_{\mathbf{0},\mathbb{R}}$.

Cf. [39] for the rest of the story.

Remark . It is interesting to note that among the people who worked on supercalculus questions, DeWitt was one of the few who did not walk into the trap of "mathematical simplicity" concerning real structures: Although in his book [14] he does not formulate the second sign rule, he actually works in the hermitian framework instead of the traditional one from the beginning. In particular, his "algebra of supernumbers" Λ_∞ is a hermitian algebra (cf. [44, 2.1] or [39]), and the "supervector space" introduced in [14, 1.4] is the same as a free hermitian module over Λ_∞.

Also, the book [18], being concerned with both classical and quantum aspects of ghosts and physical fermions, uses in fact a hermitian approach.

1.6. **Sketch of the deWitt-Rogers approach.** The difficulties mentioned motivated physicists to look for an alternative approach to the mathematical implementation of supermanifolds. This approach was pioneered by B. deWitt [14] and A. Rogers [32], and it was followed (and often technically modified) by several authors (cf. e.g. [20], [50]). Cf. the book [9] for more on the history.

The basic idea consists in realizing $m|n$-dimensional superspace as the topological space $B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n$ where $B = B_{\mathbf{0}} \oplus B_{\mathbf{1}}$ is a suitable topological $\mathbb{Z}_2$-commutative algebra. Suitability is defined differently in each version of the theory, but for the present discussion, the differences do not matter much. Here we follow the original paper of Rogers [32], that is, $B = \Lambda[\beta_1, \dots \beta_k]$ is a finite-dimensional Grassmann algebra. Equipping B with the norm $\|\sum c_\mu \beta^\mu\| := \sum |c_\mu|$, it becomes a Banach algebra.

(We note that, from the point of view of the Berezin approach, it looks suspicious that odd things appear as geometrical points. One should really work better with the purely even object $B_{\mathbf{0}}^m \times (\Pi B_{\mathbf{1}})^n$ where Π is the parity shift symbol. In this form, $m|n$-dimensional superspace will naturally appear also in 1.7 below.)

Now a map $f : U \to B$ where $U \subseteq B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n$ is open is called *superdifferentiable* iff there exist maps $D_i f : U \to B$ such that

$$\lim \frac{\|f(x+y) - f(x) - \sum y_i D_i f(x)\|}{\|y\|} = 0$$

for $x = (x_1, \dots, x_{m+n}) \in U$ and $y \to 0$ in $B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n$.

The problem with this definition is that, while the even partial derivatives $D_1 f, \dots, D_k f$ are uniquely determined, the odd ones, $D_{m+1} f, \dots, D_{m+n} f$ are determined only up to a summand of the form $g(x)\beta_1 \cdots \beta_k$ where the map $g : U \to \mathbb{R}$ is arbitrary (!). This is the first appearance of the "truncation effects" which are typical for finite-dimensional B. They can be avoided by using infinite-dimensional Grassmann algebras, but only at the price of other technical and conceptual complications; so we stick here to $k < \infty$.

Let $G^\infty(U)$ be the algebra of all infinitely often superdifferentiable functions on U. More exactly, one defines inductively $G^l(U)$ by letting $G^0(U)$ be just the set of all continuous maps $U \to B$, and $G^{l+1}(U)$ be the set of all $f : U \to B$ which are superdifferentiable such that all $D_i f$ can be choosen in $G^l(U)$; finally, $G^\infty(U) := \bigcap_{l>0} G^l(U)$.

In an obvious way, $G^\infty(U)$ is a $\mathbb{Z}_2$-graded commutative algebra over B.

Note that since B is a 2^k-dimensional vector space, f encodes 2^k real-valued functions on $\mathbb{R}^N$ with $N := 2^{k-1}(m+n)$; and superdifferentiability requires not only that these functions are differentiable but that their derivatives satisfy a certain linear system of relations, somewhat analogous to the Cauchy-Riemann relations in function theory. The expansion (1.6.1) below is a consequence of these relations.

The most obvious example for a superdifferentiable function is the projection $x_i : B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n \to B$ from the i-th factor; $x_i \in G^\infty(U)$ will play the rôle of the i-th coordinate. Sometimes we will have to treat the even and the odd ones separately; so we will also write $(u_1, \dots, u_m, \xi_1, \dots, \xi_n) := (x_1, \dots, x_{m+n})$.

Now let $\epsilon : B \to \mathbb{R}$ be the unique algebra homomorphism; thus, for every $a \in B$, its "soul" $a - \epsilon(a)$ is nilpotent, and a is invertible iff its "body", the number $\epsilon(a)$, is non-zero.

Remark . In the various variants of the theory, one has always such a unique body projection; however, if B is an infinite-dimensional algebra, the soul may be only topologically nilpotent.

We get a "body projection" from to superspace to ordinary space:

$$\epsilon : B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n \to \mathbb{R}^m, \quad (u_1, \ldots, u_m, \xi_1, \ldots, \xi_n) \mapsto (\epsilon(u_1), \ldots, \epsilon(u_m)) .$$

Let $U \subseteq B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n$ be open. Given $f \in C^\infty(\epsilon(U)) \otimes B$, i. e. a B-valued smooth function on $\epsilon(U)$, we get by "Grassmann analytic continuation" an element $z(f) \in G^\infty(U)$,

$$z(f)(u) := \sum_{\nu_1,\ldots,\nu_m \geq 0} \partial_1^{\nu_1} \cdots \partial_m^{\nu_m} f(\epsilon(u)) \frac{(u_1 - \epsilon(u_1))^{\nu_1} \cdots (u_m - \epsilon(u_m))^{\nu_1}}{\nu_1! \cdots \nu_m!};$$

the sum is actually finite. One gets an injective algebra homomorphism $z : C^\infty(\epsilon(U)) \to G^\infty(U)$, and it turns out that $G^\infty(U)$ is generated as algebra by the union of the image of this homomorphism with the set of coordinates $\{x_1, \ldots, x_{m+n}\}$. Indeed, one finds that every $f \in G^\infty(U)$ admits an expansion

$$f(u|\xi) = \sum z(f_{\mu_1 \ldots \mu_n})(u) \xi_1^{\mu_1} \cdots \xi_n^{\mu_n} = \sum z(f_\mu)(u) \xi^\mu \tag{1.6.1}$$

which not accidentally looks like (1.3.2), but here with suitable $f_{\mu_1 \ldots \mu_n} \in C^\infty(\epsilon(U))$. Unfortunately, here the truncation effects make their second appearance: the $f_{\mu_1 \ldots \mu_n}$ are uniquely determined only for $\mu_1 + \cdots + \mu_n \leq k$; otherwise, they are completely undetermined since $\xi_1^{\mu_1} \cdots \xi_n^{\mu_n}$ vanishes anyway.

Thus, we get an epimorphism of $\mathbb{Z}_2$-graded algebras

$$C^\infty(\epsilon(U)) \otimes B \otimes \Lambda[\xi_1, \ldots, \xi_m] \to G^\infty(U), \quad f \otimes \xi_1^{\mu_1} \cdots \xi_n^{\mu_n} \mapsto z(f_{\mu_1 \ldots \mu_n}) \xi_1^{\mu_1} \cdots \xi_n^{\mu_n}. \tag{1.6.2}$$

Its kernel is generated by all products $\xi_1^{\mu_1} \cdots \xi_n^{\mu_n}$ with $\mu_1 + \cdots + \mu_n > k$; in particular, (1.6.2) is an isomorphism iff $n \leq k$.

A map $B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n \supseteq U \to B_{\mathbf{0}}^{m'} \times B_{\mathbf{1}}^{n'}$, where U is open in the natural topology, is called G^∞ iff its $m' + n'$ component maps are G^∞. Now, globalizing in a standard way, one gets the notion of a G^∞ *supermanifold* which is a paracompact Hausdorff space together with an atlas of $B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n$-valued charts such that the transition maps are G^∞.

On such a G^∞ smf, there lives the sheaf $G^\infty(\cdot)$ of B-valued functions of class G^∞; one could also define a G^∞ smf equivalently as a paracompact Hausdorff ringed space which is locally isomorphic to the model space $(B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n, G^\infty(\cdot))$.

Now there exists a second reasonable notion of superfunctions and supermanifolds: Let $H^\infty(U)$ be the subalgebra of all $f \in G^\infty(U)$ for which the $f_{\mu_1 \ldots \mu_n}$ can be choosen $\mathbb{R}$-valued. This is still a $\mathbb{Z}_2$-graded commutative algebra over $\mathbb{R}$ but not over B, i. e. there are no longer anticommuting constants. Now (1.6.2) restricts to an algebra epimorphism

$$C^\infty(\epsilon(U)) \otimes \Lambda[\xi_1, \ldots, \xi_n] \to H^\infty(U). \tag{1.6.3}$$

Using H^∞ transition maps instead of G^∞ ones one gets the notion of a H^∞ *smf*. On a H^∞ smf, there lives the sheaf $H^\infty(\cdot)$ which consists of B-valued functions. However, it is no longer a sheaf of algebras over B because $H^\infty(U) \cap B = \mathbb{R}$ within $G^\infty(U)$ for any open U.

One could also define a H^∞ smf equivalently as a paracompact Hausdorff ringed space which is locally isomorphic to the model space $(B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n, H^\infty(\cdot))$.

Now every H^∞ supermanifold is also a G^∞ supermanifold in a natural way but not conversely; for the sheaves one finds $G^\infty(\cdot) \cong H^\infty(\cdot) \otimes B$.

A further ramification arises if one uses instead of the natural topology on $B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n$ the *deWitt topology* which is the inverse image topology arising from $\epsilon : B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n \to \mathbb{R}^m$; note that it is not Hausdorff. However, although it is much coarser than the natural topology, the loss of information is

smaller than one would think: It follows from the expansion (1.6.1) that once $U \subseteq V$ are open in the natural topology with $\epsilon(U) = \epsilon(V)$ the restriction maps $G^\infty(V) \to G^\infty(U)$ and $H^\infty(V) \to H^\infty(U)$ are bijective.

So one defines a *H^∞-deWitt supermanifold* as a paracompact space together with an atlas of $B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n$-valued charts, which are now local homeomorphisms w. r. to the deWitt topology, and the transition maps are H^∞. Indeed, we will see that the smf arising from a Berezin smf is naturally of this type.

1.7. **From Berezin to deWitt-Rogers.** The formal similarity of (1.6.1) to (1.3.2) suggests the existence of a connection between both approaches. Indeed, it is possible to assign to any Berezin smf a Rogers H^∞ smf which, however, will depend also on the choice of the Grassmann algebra of constants B.

While [32] gave a chart-by chart construction, we give here, following [2], a more intrinsic mechanism. It will show that the Berezin smfs are the fundamental ones; moreover, it will be in some sense prototypical for the connection of the naive notion of configurations to our one.

We first recall some well-known higher nonsense from category theory which today has become a basic tool in algebraic geometry. Given a category $\mathbf{C}$, every object X generates a cofunctor (=contravariant functor)

$$X(\cdot) : \mathbf{C} \to \mathbf{Sets}, \quad Z \mapsto X(Z)$$

where $X(Z)$ is simply the set of all morphisms from Z to X. Also, every morphism $X \to X'$ generates a natural transformation $X(\cdot) \to X'(\cdot)$. Thus, we get a functor

$$\mathbf{C} \to \{\text{category of cofunctors } \mathbf{C} \to \mathbf{Sets}\}, \quad X \mapsto X(\cdot), \tag{1.7.1}$$

and it is a remarkable observation that this functor is faithfully full, that is, every natural transformation $X(\cdot) \to X'(\cdot)$ is generated by a unique morphism $X \to X'$.

Moreover, let be given some cofunctor $F : \mathbf{C} \to \mathbf{Sets}$. If there exists an object X of $\mathbf{C}$ such that F is isomorphic with $X(\cdot)$ then this object is uniquely determined up to isomorphism; thus, one can use cofunctors to characterize objects.

In algebraic geometry, one often calls the elements of $X(Z)$ the *Z-valued points of X*. Indeed, an algebraic manifold X usually encodes a system of equations in affine or projective space which cut it out, and if Z happens to be the spectrum of a ring, $Z = \mathrm{Spec}(R)$, then $X(Z)$ is just the set of solutions of this system with values in R. One also writes simply $X(R)$ for this.

Now fix a finite-dimensional Grassmann algebra $B = \Lambda_k$. This is just the algebra of global superfunctions on the linear smf $\mathrm{L}(\mathbb{R}^{0|k})$; the underlying manifold is simply a point. $\mathrm{L}(\mathbb{R}^{0|k})$ should be viewed as C^∞ super variant of $\mathrm{Spec}(B)$, and therefore it is natural to denote, for any Berezin-Leites-Kostant smf X, by $X(B)$ the set of all smf morphisms $\mathrm{L}(\mathbb{R}^{0|k}) \to X$.

The set $X(B)$ will be the underlying set of the H^∞ smf we are going to construct. In fact, using Thm. 1.3.1, any such morphism is uniquely characterized by the pullback of global superfunctions; thus, $X(B)$ identifies with the set of all algebra homomorphisms $\mathcal{O}_\infty(X) \to B$.

Given a morphism $f : X \to Y$ of Berezin smfs, we get a map $f : X(B) \to Y(B)$, $u \mapsto u \circ f^*$. In particular, for open $U \subseteq X$, we get an injective map $U(B) \to X(B)$ which we will view as inclusion. The $U(B)$ are the open sets of a non-Hausdorff topology which is just a global variant of the deWitt topology mentioned above.

Now fix a superchart on X, i. e. an isomorphism $U \xrightarrow{\phi} U'$ where U is open in X and U' is open in $\mathrm{L}(\mathbb{R}^{m|n})_{x|\xi}$. We get a map $U(B) \xrightarrow{\phi} U'(B)$; the target of this map identifies naturally with a "Rogers superdomain":

$$U'(B) \cong \epsilon^{-1}(U') \subseteq B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n, \qquad f \mapsto (f(x_1), \ldots, f(x_m), f(\xi_1), \ldots, f(\xi_n)).$$

Denote the composite $U(B) \xrightarrow{\phi} U'(B) \to \epsilon^{-1}(U')$ by c_ϕ; it will become a superchart on $X(B)$.

Equip $X(B)$ with the strongest topology such that all c_ϕ arising from all possible supercharts on X are continuous (of course, an atlas is sufficient, too). It follows that, since the transition map between any two such supercharts is H^∞, these supercharts equip $X(B)$ with the structure of a H^∞ smf, and thus, a fortiori, a G^∞ smf.

Also, the epimorphism (1.6.3) globalizes: For $f \in \mathcal{O}_\infty(U)$ with open $U \subseteq X$, we get a map $f' : U(B) \to B, \quad u \mapsto u(f)$, which is a H^∞ superfunction. One gets an algebra epimorphism $\mathcal{O}_\infty(U) \to H^\infty(U(B))$ which is an isomorphism iff $n \geq k$.

Finally, given a morphism $f : X \to Y$ of Berezin smfs, the map $f : X(B) \to Y(B)$ considered above is of class H^∞.

Altogether, we get for fixed B a functor

(1.7.2) $$\textbf{Ber-Smfs} = \{\text{category of Berezin smfs}\} \to \{\text{category of } H^\infty \text{ smfs}\}, \quad X \mapsto X(B).$$

In particular, any ordinary smooth manifold X can be viewed as a Berezin smf and therefore gives rise to a H^∞ smf $X(B)$. Roughly speaking, while X is glued together from open pieces of $\mathbb{R}^m$ with smooth transition maps, $X(B)$ arises by replacing the open piece $U \subseteq \mathbb{R}^m$ by $\epsilon^{-1}(U) \subseteq B_{\mathbf{0}}^m$, and the transition maps by their Grassmann analytic continuations.

Remarks . (1) Apart from the fact that B is supercommutative instead of commutative, the functor (1.7.2) is just the Weil functor considered in [22].

(2) Looking at the Z-valued points of an smf is often a useful technique even if one stays in the Berezin framework. For instance, a Lie supergroup G, i. e. a group object in the category of supermanifolds, turns in this way to a functor with values in the category of ordinary groups (by the way, this property is not shared by quantum groups, which makes them much more difficult objects). While, using the diagrammatic definition of a group object, it is by no means trivial that the square of the inversion morphism $\iota : G \to G$ is the identity, this fact becomes an obvious corollary of the corresponding property of ordinary groups by using Z-valued points. Cf. [36], [25].

1.8. **Molotkov's approach and the rôle of the algebra of constants.** For a review of Molotkov's approach [29] to infinite-dimensional supermanifolds, and a comparison with our one, cf. [42]. Here we give an interpretation of the idea in categorial terms.

Roughly said, this approach relies on the description of smfs by their points with values in $0|k$-dimensional smfs, where, however, all k's are considered simultaneously.

The basic observation to understand Molotkov's approach is the following: Let **Ber-Smfs** be the category of all Berezin smfs, and let **Gr** be the category the objects of which are the real Grassman algebras $\Lambda_k = \Lambda[\xi_1, \ldots, \xi_k]$ (one for each $k \geq 0$), with all even algebra homomorphisms as morphisms. Assigning to every Λ_k the $0|k$-dimensional smf $\mathrm{Spec}(B) := \mathrm{L}(\mathbb{R}^{0|k}) = (\mathrm{point}, \Lambda_k)$ we get a cofunctor $\mathrm{Spec} : \mathbf{Gr} \to \textbf{Ber-Smfs}$ which establishes an equivalence of the opposite category $\mathbf{Gr}^{\mathrm{Op}}$ with the full subcategory $\mathrm{Spec}(\mathbf{Gr})$ of **Ber-Smfs** of those smfs which are connected and have even dimension zero.

On the other hand, it is easy to see that the subcategory $\mathrm{Spec}(\mathbf{Gr})$ is sufficient to separate morphisms, that is, given two distinct smf morphisms $X \rightrightarrows X'$ there exists some $k > 0$ and a morphism $\mathrm{Spec}(\Lambda_k) \to X$ so that the composites $\mathrm{Spec}(\Lambda_k) \to X \rightrightarrows X'$ are still different.

Thus, if assigning to any smf X the composite functor

$$\mathbf{Gr} \xrightarrow{\mathrm{Spec}} \textbf{Ber-Smfs} \xrightarrow{X(\cdot)} \mathbf{Sets},$$

we get a functor

(1.8.1) $$\textbf{Ber-Smfs} \to \{\text{category of functors } \mathbf{Gr} \to \mathbf{Sets}\}, \quad X \mapsto X(\cdot),$$

and, due to the separation property mentioned above, this functor is still faithful, i. e. injective on morphisms. (It is unlikely that it is full, but I do not know a counterexample.) Thus, an smf is characterized up to isomorphism by the functor $\mathbf{Gr} \to \mathbf{Sets}$ it generates.

However, there exist functors not representable by smfs. Roughly said, a given functor $F : \mathbf{Gr} \to \mathbf{Sets}$ is generated by an smf iff it can be covered (in an obvious sense) by "supercharts", i. e. subfunctors which are represented by superdomains. However, as a consequence of the lack of fullness of the functor (1.8.1), we cannot guarantee a priori that the transition between two supercharts is induced by a morphism of superdomains – we simply have to require that.

Explicitly, given a linear smf $\mathrm{L}(E)$ where $E \cong \mathbb{R}^{m|n}$ is a finite-dimensional $\mathbb{Z}_2$-graded vector space, one has for each $B := \Lambda_k$

$$\mathrm{L}(E)(B) = (B \otimes E)_{\mathbf{0}} \cong B_{\mathbf{0}}^m \times B_{\mathbf{1}}^n;$$

we encountered this already in the previous section.

Now, Molotkov's idea is to replace here E by a Banach space and define just the linear smf generated by E as the functor $F_E : \mathbf{Gr} \to \mathbf{Sets}$, $B \mapsto (B \otimes E)_{\mathbf{0}}$. He then defines a "superregion" alias superdomain as an in a suitable sense open subfunctor of this, and an smf as a functor $F : \mathbf{Gr} \to \mathbf{Sets}$ together with an atlas of subfunctors such that the transition between two of them is supersmooth in a suitable sense.

Actually, his functors have their values not in the set category but in the category of smooth Banach manifolds, but this is a structure which cares for itself, thanks to the use of atlasses.

Resuming we find that

- the Berezin-Kostant approach works without any "algebra of constants" except $\mathbb{R}$ or $\mathbb{C}$, i. e. without an auxiliary Grassmann algebra;
- the deWitt-Rogers approach works with a fixed Grassmann or Grassmann-like algebra of constants;
- Molotkov uses all finite-dimensional Grassmann algebras at the same time as algebras of constants, and thus achieves independence of a particular choice.

My personal point of view is that any differential-geometric concept which makes explicit use of the algebra of constants B, and hence is not functorial w. r. to a change of B, is suspicious. Although the possibility of choice of a concrete B allows new differential-geometric structures and phenomena, which by some people are claimed to be physically interesting, there is up to now no convincing argument for their potential physical relevance.

In particular, this is connected with the fact that there exist Rogers smfs which do not arise from a Berezin smf by the construction given above. A prototypical example is a $1|1$-dimensional supertorus, in which also the odd coordinate is wrapped around. It is clearly non-functorial in B because it needs the distinction of a discrete subgroup in $B_{\mathbf{1}}$.

Also, if one defines G^∞ Lie supergroups in the obvious way, the associated Lie superalgebra will be a module over B, with the bracket being B-linear. It follows that the definition is bound to a particular B.

In particular, if G is a Lie supergroup in Berezin's sense, with Lie superalgebra $\mathfrak{g}$, then, for a Grassmann algebra B, the Rogers Lie supergroup $G(B)$ will have Lie superalgebra $\mathfrak{g} \otimes B$; Now it is an abundance of Lie superalgebras over B which are not of this form; cf. [14, 4.1] for an example of a $0|1$-dimensional simple Lie superalgebra which cannot be represented as $\mathfrak{g} \otimes B$. However, none of these "unconventional" Lie superalgebras seems to be physically relevant.

A general metamathematical principle, which is up to now supported by experience, is: Any mathematical structure which does not make *explicit* use of the algebra of constants B can be (and should be) formulated also in Berezin terms.

In particular, this applies to the structures appearing in geometric models in quantum field theory.

1.9. **Comparison of the approaches.** The deWitt-Rogers approach has quite a few psychological advantages: Superfunctions are now B-valued maps, i. e. genuine functions on this space instead of elements of an abstract algebra. Also, instead of morphisms of ringed spaces one has genuine superdifferentiable maps.

Also, from an aesthetic point of view, even and odd degrees of freedom seem to stand much more on equal footing: While the underlying space of a Berezin smf encodes only even degrees of freedom, the underlying space of a Rogers-like smf is a direct product of even and odd part.

Finally, in the Taylor expansion (1.6.1) of a G^∞ function at $\xi = 0$, the functions $z(f_\mu)(u)$ are still B-valued, and thus they may perfectly well anticommute (they do so iff they are $B_{\mathbf{1}}$-valued). At the first glance, this approach appears to be ideally suited for the implementation of the anticommutativity of fermionic field components.

These appealing features tempted quite a few physicists to prefer this approach to the Berezin approach (cf. in particular [14]). But one has to pay for them: "This description, however, is extremely uneconomical because in the analysis do the ξ's" (here θ's) "explicitly enter. So, the ξ's are really unnecessary" ([18, 6.5.3]).

For instance, the 4|4-dimensional super Minkowski space, that is, the space-time of e.g. $N = 1$ super Yang-Mills theory with superfields, is modelled as $X = B_{\mathbf{0}}^4 \times B_{\mathbf{1}}^4$. Thus, one has an "inflation of points", and even the most eager advocates of this approach have no idea how to distinguish all these points physically. In fact, the algebra B plays an auxiliary role only. Notwithstanding which of the modern, mainly heuristic, concepts of quantization of a field theory one uses, B does not appear at all at the quantized level. In other words, B is an unphysical "addendum" used to formulate the classical model. This becomes even clearer if we try to model fermion fields on the ordinary Minkowski space within this approach: Space-time would be described by $B_{\mathbf{0}}^4$ instead of the usual (and certainly more appropriate) $\mathbb{R}^4$.

On the other hand, one can obtain anticommuting field components necessary for modelling fermion fields also in the framework of Berezin's approach: Instead of real- or complex-valued functions or superfunctions $f \in \mathcal{O}_\infty(X)$ one uses "B-valued" ones: $f \in \mathcal{O}_\infty(X) \otimes_{\mathbb{R}} B$ where B is a Grassmann algebra or some other suitable $\mathbb{Z}_2$-commutative algebra, and the tensor product is to be completed in a suitable sense if B is infinite-dimensional.

This removes the inflation of points – but the unphysical auxiliary algebra B remains. At any rate, this argument shows that the deWitt-Rogers approach has not the principal superiority over Berezin's approach which is sometimes claimed by its advocates. The problem of "B-valued configurations" will be discussed in the next section.

The question arises, what algebra of constants B should we take, how "large" should it be? As we saw in 1.6, in particular, in considering the epimorphism (1.6.2), a finite-dimensional B leads to various unwanted truncation effects; these can be avoided by taking B infinite-dimensional. But even this decision leaves still room; instead of a Banach completion of a Grassmann algebra with a countable set of generators, one could use e.g. one of the Banach algebras $\mathcal{P}(p; \mathbb{R})$ introduced in [44] (with a purely odd field target space).

On the other hand, an $m|n$-dimensional smf will now become as topological space $m \cdot \dim(B_{\mathbf{0}}) + n \cdot \dim(B_{\mathbf{1}})$-dimensional. One feels uneasy with such an inflation of dimensions. Moreover, we cannot use the manifolds of ordinary differential geometry directly – we have to associate to them the corresponding smfs (although it is an almost everywhere trivial process to "tensor through" ordinary structures to their B-analogues, for instance, this applies to metrics, connections, vector bundles etc).

Still worse, the automorphism group $\mathrm{Aut}_{\mathbb{R}}(B)$, which is at least for Grassmann algebras pretty big, appears as global symmetry group; thus, we have *spurious symmetries*. Indeed, given a Lagrange density which is a differential polynomial $\mathcal{L} \in \mathbb{C}\left\langle \underline{\Phi} | \underline{\Psi} \right\rangle$ and an automorphism $\alpha : B \to B$, the field equations will stay invariant under $(\Phi, \Psi) \mapsto (\alpha\Phi, \alpha\Psi)$. (This is analogous to the process in which the

only continuous automorphism of $\mathbb{C}$, the conjugation, induces the charge conjugation for a complex field.) Of course, this symmetry could be inhibited by using a differential polynomial with coefficients in B, i. e. $\mathcal{L} \in B\langle\underline{\Phi}|\underline{\Psi}\rangle$, but all usual models have coefficients in $\mathbb{C}$.

One feels that the algebra of constants B is an addendum, something which is not really a part of the structure, and that it should be thrown out. This impression is still stronger if we look at a quantized theory: Sometimes, for instance in [14], it is claimed that the result of quantization should be a a "super Hilbert space", i. e. a B-module $\mathcal{H}$ together with a skew-linear B-valued scalar product $\langle\cdot|\cdot\rangle$ such that the body of $\langle\phi|\phi\rangle$ for $\phi \in \mathcal{H}$ is non-negative, and is positive iff the body of ϕ is non-vanishing.

However, no one has ever measured a Grassmann number, everyone measures real numbers. Moreover, while a bosonic field strength operator $\widehat{\Phi}(x)$ (or, more exactly, the smeared variant $\int dx\, g(x)\widehat{\Phi}(x)$) encodes in principle an observable (apart from gauge fields, where only gauge-invariant expressions are observable), a fermionic field strength operator $\widehat{\Psi}(x)$ (or $\int dx\, g(x)\widehat{\Psi}(x)$) is only a "building block" for observables. That is, the eigenvalues of $\int dx\, g(x)\widehat{\Psi}(x)$ do not have a physical meaning.

So it is in accordance with the quantized picture to assume on the classical level that bosonic degrees of freedom are real-valued while the fermionic ones do not take constant values at all. (In fact, the picture is somewhat disturbed by the fact that, in the quantized picture, multilinear combinations of an even number of fermion fields do take values, like e. g. the electron current, or the pion field strength; that is, although classical fermion fields have only "infinitesimal" geometry, their quantizations generate non-trivial, non-infinitesimal geometry.)

1.10. **B-valued configurations.** Suppose we have a classical field model on $\mathbb{R}^4$. Usually, the theory is described by saying what boson fields $\Phi(\cdot)$ and fermion fields $\Psi(\cdot)$ appear, and a Lagrange density $\mathcal{L}[\Phi, \Psi]$, which, at the first glance, is a functional of these fields.

In the language of [44], the model is given by fixing a field target space V, which is a finite-dimensional $\mathbb{Z}_2$-graded vector space on which the two-fold cover of the Lorentz group acts, and a differential polynomial $\mathcal{L}[\underline{\Phi}, \underline{\Psi}] \in \mathbb{C}\langle\underline{\Phi}|\underline{\Psi}\rangle$.

The most obvious way to implement (1.1.1) is to assume that the fermion fields $\Psi(\cdot)$ are functions on $\mathbb{R}^4$ with values in the odd part of a $\mathbb{Z}_2$-graded commutative algebra B, for instance, a Grassmann algebra Λ_n.

Now, if one takes the field equations serious (indeed, they should give the first order approximation of the quantum dynamics) then it follows that boson fields cannot stay any more real- or complex-valued; instead, they should have values in the even part of B. Cf. e. g. [10] for a discussion of the classical field equations of supergravity within such an approach.

Also, if the model under consideration works with superfields, then these should be implemented as B-valued superfunctions. For instance, the chiral superfield (1.1.2) is now implemented as an element $\Phi \in \left(\mathcal{O}_\infty(\mathrm{L}(\mathbb{R}^{4|4})) \otimes B\right)_{\mathbf{0}}$ which satisfies the constraint $\overline{D}_{\dot\alpha}\Phi = 0$; the requirement that Φ be even leads to the required anticommutativity of the Weyl spinor $\psi \in C^\infty(\mathbb{R}^4, \mathbb{C}^2) \otimes B_{\mathbf{1}}$.

Now there are arguments which indicate that the "B-valued configurations" should not be the ultimate solution of the problem of modelling fermion fields: First, as in 1.9, it is again not clear which algebra of constants B one should take.

Also, any bosonic field component will have $\dim(B_{\mathbf{0}})$ real degrees of freedom instead of the expected single one. Thus, we have *fake degrees of freedom*, quite analogous to the "inflation of points" observed in 1.9.

Our remarks in 1.9 on *spurious symmetries* as well as on the inobservability of Grassmann numbers apply also here.

Also, the $B_{\mathbf{0}}$-valuedness of the bosonic fields makes their geometric interpretation problematic. Of course, it is still possible to interpret a $B_{\mathbf{0}}$-valued gauge field as $B_{\mathbf{0}}$-linear connection in a bundle of $B_{\mathbf{0}}$-modules over $\mathbb{R}^4$ (instead of an ordinary vector bundle, as it is common use in instanton theory). But what is the geometric meaning of a $B_{\mathbf{0}}$-valued metric $g_{\mu\nu}$?

For every finite-dimensional B there exists some N such that the product of any N odd elements vanishes, and this has as consequence the unphysical *fake relation*

$$\Psi_{j_1}(x_1)\cdots\Psi_{j_N}(x_N) = 0.$$

This fake relation can be eliminated by using an infinite Grassmann algebra, like e. g. the deWitt algebra Λ_∞, or some completion of it; but the price to be paid is that we now use infinitely many real degrees of freedom in order to describe just one physical d. o. f.

Finally, in the functional integral

$$\langle \hat{A} \rangle_{\text{vac}} = \frac{1}{N}\int [D\Phi][D\overline{\Psi}][D\Psi]\exp(\mathrm{i}\,S[\Phi,\overline{\Psi},\Psi]/\hbar)A[\Phi,\overline{\Psi},\Psi]$$

– however mathematically ill-defined it may be – the bosonic measure $[D\Phi]$ still runs (or is thought to do so) over $\mathbb{R}$-valued configurations, not over $B_{\mathbf{0}}$-valued ones.

All these arguments indicate that B-valued configurations should not be the ultimate solution to the problem of modelling fermionic degrees of freedom. Nevertheless, we will see in a moment that they do have a satisfactory geometric interpretation, at least if $B = \Lambda_k$ is a finite-dimensional Grassmann algebra: They should be thought of as *families of configurations* parametrized by the $0|k$-dimensional smf $\mathrm{L}(\mathbb{R}^{0|k})$.

1.11. **Families of configurations.** Historically, the "family philosophy" comes from algebraic geometry; it provides a language for the consideration of objects varying algebraically (or analytically) with parameters. It turns out to be rather useful in supergeometry, too; cf. [25], [36], [42].

We begin with the standard scalar field theory given by the Lagrangian density

$$\mathcal{L}[\underline{\Phi}] = \sum_{a=0}^{3} \partial_a\underline{\Phi}\partial^a\underline{\Phi} + m^2\underline{\Phi}^2 + V(\underline{\Phi})$$

where V is a polynomial, e. g. $V(\underline{\Phi}) = \underline{\Phi}^4$.

Thus, a configuration is simply a real function on $\mathbb{R}^4$; for convenience, we will use smooth functions, and the space of all configurations is the locally convex space $M = C^\infty(\mathbb{R}^4)$. For any bounded open region $\Omega \subseteq \mathbb{R}^4$, the action on Ω is a real-analytic function

$$S_\Omega[\cdot] : M \to \mathbb{R}, \quad \phi \mapsto S_\Omega[\phi] := \int_\Omega d^4x\mathcal{L}[\phi]. \tag{1.11.1}$$

Now suppose that $\phi = \phi(x_0,\ldots,x_3|\zeta_1,\ldots,\zeta_n)$ depends not only on the space-time variables $x_0,\ldots,x_3$ but additionally on odd, anticommuting parameters $\zeta_1,\ldots,\zeta_n$. In other words, $\phi \in \mathcal{O}_\infty(\mathbb{R}^4 \times \mathrm{L}(\mathbb{R}^{0|n}))_{\mathbf{0},\mathbb{R}}$ is now a *family of configurations* parametrized by the $0|n$-dimensional smf $Z_n := \mathrm{L}(\mathbb{R}^{0|n})$ (we require ϕ to be even in order to have it commute).

As in (1.3.2) we may expand $\phi(x|\zeta) = \sum_{\mu\in\mathbb{Z}_2^n,\ |\mu|\equiv 0(2)} \phi_\mu(x)\zeta^\mu$, i. e. we may view ϕ as a smooth map

$$\phi : \mathbb{R}^4 \to \mathbb{C}[\zeta_1,\ldots,\zeta_n]_{\mathbf{0},\mathbb{R}}.$$

That is, ϕ is nothing but a B-valued configuration with the Grassmann algebra $B := \mathbb{C}[\zeta_1,\ldots,\zeta_n]$ as auxiliary algebra!

Encouraged by this, we look at a model with fermion fields, say

$$\mathcal{L}[\underline{\Phi}|\underline{\Psi}] := \frac{\mathrm{i}}{2}\sum_{a=0}^{4}\left(\overline{\underline{\Psi}}\gamma^a\partial_a\underline{\Psi} - \partial_a\overline{\underline{\Psi}}\gamma^a\underline{\Psi}\right) - m^{\Psi}\overline{\underline{\Psi}}\underline{\Psi} - \sum_{a=0}^{4}\partial_a\underline{\Phi}\partial^a\underline{\Phi} - (m^{\Phi})^2\underline{\Phi}^2 - \mathrm{i}\,g\overline{\underline{\Psi}}\underline{\Psi}\underline{\Phi} \tag{1.11.2}$$

where $\underline{\Phi}$ is a scalar field, $\underline{\Psi}$ a Dirac spinor, and the thick bar denotes the Dirac conjugate (in fact, (1.11.2) is a slightly dismantled version of the Yukawa model of meson-nucleon scattering).

Note that (1.11.2) is well-defined as a differential polynomial $\mathcal{L}[\underline{\Phi}|\underline{\Psi}] \in \mathbb{C}\left\langle\underline{\Phi}|\underline{\Psi}\right\rangle_{\mathbf{0},\mathbb{R}}$.

While a naive configuration for the fermion field, $\psi = (\psi_\alpha) \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$, would violate the requirement of anticommutativity of the field components ψ_α, this requirement is satisfied if we look for a family $(\psi_\alpha) = (\psi_\alpha(x_0, \ldots, x_3|\zeta_1, \ldots, \zeta_n))$, which now consists of odd superfunctions $\psi_\alpha \in \mathcal{O}_\infty(\mathbb{R}^4 \times \mathrm{L}(\mathbb{R}^{0|n}))_{\mathbf{1}}$. Looking again at the expansion, we can reinterpret ψ_α as a smooth map

$$\psi_\alpha : \mathbb{R}^4 \to \mathbb{C}[\zeta_1, \ldots, \zeta_n]_{\mathbf{1}},$$

that is, ψ is nothing but a B-valued configuration again.

Resuming, we can state that if $B = \mathbb{C}[\zeta_1, \ldots, \zeta_n]$ is a Grassmann algebra then *B-valued configurations can be viewed as families of configurations parametrized by the smf* $\mathrm{L}(\mathbb{R}^{0|n})$.

Now there is no need to use only $0|n$-dimensional smfs as parameter space. Thus, for the model (1.11.2), a *Z-family of configurations* where Z is now an arbitrary Berezin supermanifold, is a tuple

$$(\phi, \psi_1, \ldots, \psi_4) \in \mathcal{O}_\infty(\mathbb{R}^4 \times Z)_{\mathbf{0},\mathbb{R}} \times \prod^{4} \mathcal{O}_\infty(\mathbb{R}^4 \times Z)_{\mathbf{1}}, \tag{1.11.3}$$

and the action over any compact space-time domain $\Omega \subseteq \mathbb{R}^4$ becomes an element

$$S_\Omega[\phi|\psi] := \int_\Omega d^4x\mathcal{L}[\phi|\psi] \in \mathcal{O}_\infty(Z)_{\mathbf{0},\mathbb{R}}. \tag{1.11.4}$$

Also, if $\mu : Z' \to Z$ is some smf morphism then (1.11.3) can be pullbacked along the morphism $1_{\mathbb{R}^4} \times \mu : \mathbb{R}^4 \times Z' \to \mathbb{R}^4 \times Z$ to give a Z'-family $(\phi', \psi_1', \ldots, \psi_4')$.

The simplest case is that $Z = P$ is a point: Since $\mathcal{O}_\infty(\mathbb{R}^4 \times P) = C^\infty(\mathbb{R}^4)$, every P-family has the form $(\phi|0, 0, 0, 0)$ where $\phi \in C^\infty(\mathbb{R}^4)$. Thus it encodes simply a configuration for the bosonic sector.

More generally, if Z has odd dimension zero, i. e. is essentially an ordinary manifold, then for Z-family we have $\psi_\alpha = 0$ by evenness. Hence *we need genuine supermanifolds as parameter spaces in order to describe non-trivial configurations in the fermionic sector.*

If Z is a superdomain with coordinates $z_1, \ldots, z_m|\zeta_1, \ldots, \zeta_n$ then (1.11.3) is simply a collection of one even and four odd superfunctions which all depend on $(x_0, \ldots, x_3, z_1, \ldots, z_m|\zeta_1, \ldots, \zeta_n)$, and the pullbacked family now arises simply by substituting the coordinates $(z_i|\zeta_j)$ by their pullbacks in $\mathcal{O}_\infty(Z')$; this process is what a physicist would call a *change of parametrization.*

This suggests to look for a *universal family* from which every other family arises as pullback. This universal family would then encode all information on classical configurations.

Of course, the parameter smf M_{C^∞} of such a family, i. e. the *moduli space* for configuration families, is necessarily infinite-dimensional; hence we will have to extend the Berezin-Leites-Kostant calculus (or, strictly spoken, its hermitian variant as sketched in 1.5) to a calculus allowing $\mathbb{Z}_2$-graded locally convex spaces as model spaces; we will do so in section 2.

So, sticking to our example Lagrangian (1.11.2), we are looking for an $\infty|\infty$-dimensional smf M_{C^∞} and a tuple of smooth superfunctions

$$(\Phi, \Psi_1, \ldots, \Psi_4) \in \mathcal{O}_\infty(\mathbb{R}^4 \times M_{C^\infty})_{\mathbf{0},\mathbb{R}} \times \prod^{4} \mathcal{O}_\infty(\mathbb{R}^4 \times M_{C^\infty})_{\mathbf{1}} \tag{1.11.5}$$

such that for any other family (1.11.3) there exists a unique morphism of smfs $\mu_{(\phi,\psi_1,\ldots,\psi_4)} : Z \to M$ such that (1.11.3) is the pullback of (1.11.5) along this morphism. We call $\mu_{(\phi,\psi_1,\ldots,\psi_4)}$ the *classifying* morphism of the family (1.11.3).

However, we will not implement (1.11.5) in the verbal sense since a definition of *smooth* superfunctions in the infinite-dimensional case, which is needed to give the symbol "$\mathcal{O}_\infty$" in (1.11.5) sense, is technically rather difficult, and we will save much work by sticking to real-analytic infinite-dimensional supermanifolds.

Fortunately, this difficulty is easily circumvented: We simply consider only those families (1.11.3) for which Z is actually a real-analytic smf, and the Z-dependence is real-analytic (note that if $Z = \mathrm{L}(\mathbb{R}^{0|n})$ is purely even then this requirement is empty since we have actually polynomial dependence). Once the sheaf $\mathcal{O}^{C^\infty(\mathbb{R}^4)}(\cdot)$ of superfunctions on Z with values in the locally convex space $C^\infty(\mathbb{R}^4)$ has been defined (cf. 2.4 below), we can then rewrite (1.11.3) to an element

$$(\phi, \psi_1, \ldots, \psi_4) \in \mathcal{O}^{C^\infty(\mathbb{R}^4)}(Z)_{\mathbf{0},\mathbb{R}} \times \prod^4 \mathcal{O}^{C^\infty(\mathbb{R}^4)}(Z)_{\mathbf{1}}.$$

Comparing with Cor. 1.3.2 and its infinite-dimensional version Thm. 2.8.1 below it is now easy to see how to construct the universal family: Its parameter smf is simply a linear smf, $M_{C^\infty} = \mathrm{L}(E)$, with the $\mathbb{Z}_2$-graded locally convex space

$$E = \underbrace{C^\infty(\mathbb{R}^4)}_{\text{even part}} \oplus \underbrace{C^\infty(\mathbb{R}^4) \otimes \mathbb{C}^{0|4}}_{\text{odd part}}$$

as model space, and the universal family (1.11.5) is now just the standard coordinate on this infinite-dimensional smf:

$$(\Phi, \Psi_1, \ldots, \Psi_4) \in \mathcal{O}^{C^\infty(\mathbb{R}^4)}(M_{C^\infty})_{\mathbf{0},\mathbb{R}} \times \prod^4 \mathcal{O}^{C^\infty(\mathbb{R}^4)}(M_{C^\infty})_{\mathbf{1}}.$$

Comparing with 1.7, we see that the set of B-valued configurations stands in the same relation to our configuration smf M_{C^∞} as a finite-dimensional Berezin smf X to the associated H^∞-deWitt smf $X(B)$. That is, all information is contained in M_{C^∞}, which therefore should be treated as the fundamental object.

For instance, as special case of (1.11.4) and as supervariant of (1.11.1), the action on a bounded open region $\Omega \subseteq \mathbb{R}^4$ is now a real-analytic scalar superfunction on M:

$$S_\Omega[\Phi|\Psi] := \int_\Omega d^4x \mathcal{L}[\Phi|\Psi] \in \mathcal{O}(M_{C^\infty})_{\mathbf{0},\mathbb{R}}, \tag{1.11.6}$$

and the action (1.11.4) of any family $(\phi|\psi)$ is just the pullback of (1.11.6) along its classifying morphism. Thus, the element (1.11.6) can be called the *universal action* on Ω.

Also, the field strengthes at a space-time point $x \in \mathbb{R}^4$ become superfunctions:

$$\Phi(x) \in \mathcal{O}(M_{C^\infty})_{\mathbf{0},\mathbb{R}}, \quad \Psi_\alpha(x) \in \mathcal{O}(M_{C^\infty})_{\mathbf{1}}.$$

It is now the $\mathbb{Z}_2$-graded commutativity of $\mathcal{O}(M_{C^\infty})$ which ensures the validity of (1.1.1).

1.12. **Classical configurations, functionals of classical fields, and supermanifolds.** Here we formulate our approach programmatically; an exposition of bourbakistic rigour will follow in the successor papers [45], [46].

1.12.1. *Classical configurations.* The keystone of our approach is the following:

The configuration space of a classical field model with fermion fields is an $\infty|\infty$-dimensional supermanifold M. Its underlying manifold is the set M_{bos} of all configurations of the bosonic fields while the fermionic degrees of freedom are encoded as the odd dimensions into the structure sheaf $\mathcal{O}_M$.

The functionals of classical fields, which we described in [44] *as superfunctionals, are just the superfunctions on M. If such a functional describes an observable it is necessarily even and real.*

Suppose that our model describes V-valued fields on flat space-time $\mathbb{R}^{d+1}$, as in [44] (this is the case for almost all common models in Minkowski space; in Yang-Mills theory, it is the case after choosing a reference connection; however, it is no longer true for σ models or for models including gravitation in the usual way). In that case, the "naive configuration space" is a suitable (cf. Rem. (1) below) admissible space E with respect to the setup (d, V) (we recall that this means $\mathcal{D}(\mathbb{R}^{d+1}) \otimes V \subseteq E \subseteq \mathcal{D}'(\mathbb{R}^{d+1}) \otimes V$ with dense inclusions), and the supermanifold of configurations is just the linear smf

$$\mathrm{L}(E) := (E_{\mathbf{0}}, \mathcal{O}_E),$$

i. e. it has underlying space $E_{\mathbf{0}}$, while the structure sheaf is formed by the superfunctionals introduced in [44] (cf. 2.7 below for the precise definition of smfs). For more general models, e. g. σ models, the configuration space has to be glued together from open pieces of linear superdomains.

The flaw of the usual attempts to model fermion fields lies in the implicite assumption that the configuration space M should be a *set*, so that one can ask for the form of its elements. However, if M is a supermanifold in Berezin's sense then it has no elements besides that of its underlying manifold M_{bos}, which just correspond to configurations with all fermion fields put to zero.

However, although "individual configurations" do not exist (besides the purely bosonic ones), *families of configurations* parametrized by supermanifolds Z do exist. Precise definitions will be given in the successor papers. Here we note:

A Z-family of configurations encodes the same information as a morphism of supermanifolds $Z \to M$.

Thus, M can be understood as the representing object for the cofunctor

$$\mathbf{Smfs} \to \mathbf{Sets}, \quad Z \mapsto \{Z\text{-families of configurations}\}.$$

In particular, let us look for field configurations with values in a finite-dimensional complex Grassmann algebra $\Lambda_n := \mathbb{C}[\zeta_1, \ldots, \zeta_n]$. Since $\Lambda_n = \mathcal{O}(\mathrm{L}(\mathbb{R}^{0|n}))$, such a Λ_n-valued configuration should be viewed as $\mathrm{L}(\mathbb{R}^{0|n})$-family of configurations, and thus encodes a morphism of supermanifolds $\mathrm{L}(\mathbb{R}^{0|n}) \to M$.

Thus, the set of all Λ_n-valued configurations is in natural bijection with the set of all morphisms from $\mathrm{L}(\mathbb{R}^{0|n})$ to M. This shows that the really fundamental object is M, and it explains the arbitrariness in the choice of the "auxiliary algebra" B in any Rogers-like formulation.

With this point of view, we can do without Rogers' supermanifolds at all. Even better, one has a functor from "Berezin things" to "Rogers things": given a Berezin supermanifold X the set of all $\mathrm{L}(\mathbb{R}^{0|n})$-valued points of X forms in a natural way a Rogers supermanifold.

Remarks . (1) The precise meaning of the notion "configuration space" depends on the choice of the model space E, i. e. on the functional-analytic quality of the configurations to be allowed. If E is too large (for instance, the maximal choice is $E = \mathcal{D}'(\mathbb{R}^{d+1}) \otimes V$) then we have few classical functionals (in our example, all coefficient functions have to be smooth, and there are no local functionals besides linear ones), and the field equations may be ill-defined. On the other hand, if E is too small then we have a lot of classical functionals but possibly few classical solutions of the field equations; e.g., for $E = \mathcal{D}(\mathbb{R}^{d+1}) \otimes V$, there are no non-vanishing classical solutions.

If necessary one can consider different configuration spaces for one model; the discussion at the end of 1.4 suggests that this might be appropriate. In the case of models with linear configuration space, each choice of E defines its own configuration supermanifold $\mathrm{L}(E)$. We note that a continuous inclusion of admissible spaces, $E \subseteq E'$, induces a morphism of supermanifolds $\mathrm{L}(E) \to \mathrm{L}(E')$.

(2) The question arises whether the supermanifold M should be analytic, or smooth? Up to now, we stuck to the real-analytic case; besides of the fact that, in an infinite-dimensional situation, this is technically easier to handle, this choice is motivated by the fact that the standard observables (like

action, four-momentum, spinor currents etc.) are integrated differential polynomials and hence real-analytic superfunctionals, and so are also the field equations which cut out the solution supermanifold M^{sol} (cf. below).

Unfortunately, the action of the symmetry groups of the model will cause difficulties, as we will discuss in the successor paper [46]. Therefore it is not superfluous to remark that a C^∞ calculus of ∞-dimensional supermanifolds is possible.

Fortunately, if the supermanifold of classical solutions (cf. 1.12.2 below) exists at all, it will be real-analytic.

(3) Already B. DeWitt remarked in [13] that the configuration space of a theory with fermions should be regarded as a Riemannian supermanifold. Strangely enough, the same author wrote the book [14] in which he elaborated a Rogers-like approach with all details. Cf. also p. 230 of [14] in which he emphasizes again that the configuration space is a supermanifold C – of course, in a Rogers-like sense again. Now his algebra Λ_∞ of "supernumbers" turns out to be the algebra of global superfunctions on the $0|\infty$-dimensional smf $\mathrm{L}(\Pi\mathbb{R}^\infty)$, and C is again the set of all $\mathrm{L}(\Pi\mathbb{R}^\infty)$-valued points of M. Cf. [45] for details.

The "supermanifold of fields" M is (somewhat incidentally) mentioned also in [18, 18.1.1].

(4) Although our point of view could look crazy in the eyes of physicists, it is in perfect agreement with their heuristic methods to handle classical fermion fields. In particular, this applies to the fermionic variant of the Feynman integral over configurations: Consider the Gaussian integral

$$\int [D\overline{\Psi}][D\Psi]\exp(-\overline{\Psi}A\Psi) = \mathrm{Det}A \tag{1.12.1}$$

where Ψ is a fermion field. The usual, heuristic justification of the setting (1.12.1) amounts in fact to an analogy conclusion from the model case of the integral over a volume form over a $0|n$-dimensional complex supermanifold (cf. [3] or modern textbooks on quantum field theory.From this point of view, it is very natural to think that the left-hand side of (1.12.1) is in fact the integral of a volume form on a $0|\infty$-dimensional supermanifold M.

(5) It is interesting to note that we make contact with supergeometry whenever a classical field model contains fermion fields – irrespective of whether supersymmetries are present or not. Perhaps, it is not devious to view this as a "return to the sources". Indeed, the source of supergeometry was just the anticommutativity of classical fermion fields.

(6) Of course, the programme just being presented applies equally well to string models since the latter can be viewed as two-dimensional field models. On the other hand, we neglect all models with more complicated ("plecton" or "anyon") statistics; although they nowadays have become fashionable, they have up to now not been proven to have fundamental physical relevance for particle physics.

In order to implement our approach also for more general models, one needs a theory of infinite-dimensional supermanifolds. In [40], [41], [42], the present author constructed a general theory of infinite-dimensional real or complex analytic supermanifolds modelled over arbitrary locally convex topological vector spaces. In fact, [44] presented a specialization of elements of this theory to the case that the model space is an admissible function spaces on $\mathbb{R}^d$.

In section 2, we abstract from the function space nature of E, and we globalize the theory from superdomains to supermanifolds. Thus, we will give an alternative description of the theory mentioned above.

1.12.2. *Classical action, field equations, and classical solutions.* Turning to the action principle, we begin with a naive formulation which is suitable for Euclidian models:

The classical action S is an observable; thus, $S \in \mathcal{O}(M)_{\mathbf{0},\mathbb{R}}$. Moreover, the condition $\delta S = 0$ should cut out in M a sub-supermanifold M^{sol}, the supermanifold of the classical solutions.

Here

$$\delta S = \int_{\mathbb{R}^{d+1}} dx \,\Big(\sum_{i=1}^{N^\Phi} \frac{\delta}{\delta\Phi_i(x)} S \;\delta\Phi_i(x) + \sum_{j=1}^{N^\Psi} \frac{\delta}{\delta\Psi_j(x)} S \;\delta\Psi_j(x)\Big) \in \Omega^1(M)$$

is the total variation (cf. 3.8), alias exterior differential, of S.

The precise meaning of the phrase "cutting out" will be explained in 2.12.

Note that this naive formulation of the action principle is doomed to failure for models in Minkowski space: there is not a single nonvanishing solution of the free field equations for which the action over the whole space-time is finite. Thus, there is no element of $\mathcal{O}(M)_{\mathbf{0},\mathbb{R}}$ which describes the action over the whole space-time, and one has to use a "localized" variant of the action principle: Usually, the Lagrangian is given as a differential polynomial $\mathcal{L}[\underline{\Xi}] = \mathcal{L}[\underline{\Phi}|\underline{\Psi}]$ (cf. [44, 2.8] for the calculus of differential polynomials). What we have is the action over bounded open space-time regions $S_\Omega[\Xi] := \int_\Omega dx \mathcal{L}[\Xi](x) \in \mathcal{O}(M)_{\mathbf{0},\mathbb{R}}$, and we may call a Z-family of configurations $\Xi' = (\Phi'|\Psi')$ a *Z-family of solutions* iff $(\chi S_\Omega)[\Xi'] = 0$ for all vector fields χ on the configuration smf which have their "target support" in the interior of Ω (cf. [46] for precise definitions).

Now a Z-family of configurations Ξ' is a family of solutions iff it satisfies the arising field equations

$$\frac{\delta}{\delta \underline{\Xi}_i} \mathcal{L}[\Xi'] = 0$$

we call it a *Z-family of solutions.* Note that we take here not functional derivatives, but variational ones; they are defined purely algebraically.

Now, recalling that a Z-family Ξ' is in essence the same as a morphism $\breve{\Xi}' : Z \to M$, the smf of classical solutions will be characterized by the following "universal property":

A Z-family of configurations Ξ' is a family of solutions iff the corresponding morphism $\breve{\Xi}' : Z \to M$ factors through the sub-smf M^{sol}.

It follows that M^{sol} can be understood as the representing object for the cofunctor

$$\mathbf{Smfs} \to \mathbf{Sets}, \quad Z \mapsto \{Z\text{-families of solutions}\}.$$

Roughly spoken, the superfunctions on M^{sol} are just the classical *on-shell observables* (i. e. equivalence classes of observables with two observables being equivalent iff they differ only by the field equations).

In the context of a model in Minkowski space, it is a natural idea that any solution should be known by its Cauchy data, and that the "general" solution, i. e. the universal solution family to be determined, should be parametrized by all possible Cauchy data.

Thus, one introduces an *smf of Cauchy data* M^{Cau} the underlying manifold of which will be the manifold of ordinary Cauchy data for the bosonic fields, and one constructs an smf morphism

$$\Xi^{\mathrm{sol}} : M^{\mathrm{Cau}} \to M \tag{1.12.2}$$

with the following property: Given a morphism $\zeta : Z \to M^{\mathrm{Cau}}$, i. e. effectively a Z-family of Cauchy data, the composite $\Xi^{\mathrm{sol}} \circ \zeta : Z \to M$ describes the unique Z-family of solutions with these Cauchy data. Thus, Ξ^{sol} is now the *universal family of solutions* from which every other family of solutions arises as pullback. The image of the morphism (1.12.2) will be just the sub-smf M^{sol} of solutions.

The successor papers [45], [46] will implement this point of view.

1.12.3. *Symmetries of the classical theory.* First, let us look at Euclidian models, where the global action exists as a superfunction $S \in \mathcal{O}(M)_{\mathbf{0},\mathbb{R}}$:

The (super) Lie algebra of the infinitesimal symmetries of the theory manifests itself as (super) Lie algebra of vector fields on M which leave S invariant. These vector fields are tangential to M^{sol} and hence restrict to it.

The discussion of [44, 3.13] yields a example for the appearance of supersymmetry as an algebra of vector fields on a configuration supermanifold (which, however, is so small that it allows a globally defined action but no non-trivial solutions of the field equations).

On the other hand, *on-shell (super-) symmetry manifests itself as linear space of vector fields on M which still leave S invariant, but which only on M^{sol} form a representation of the super Lie algebra of supersymmetry.*

Unfortunately, in the Minkowski situation, the non-existence of a globally defined action makes it difficult to give a general definition of symmetries of the action; we will discuss that elsewhere.

Naively, one should expect that any infinitesimal (super-) symmetry algebra integrates to an action $\alpha : G \times M \to M$ of the corresponding connected, simply connected (super) Lie group G on M. For Yang-Mills gauge symmetry, this will work perfectly well; on the other hand, for space-time symmetry groups, there will arise the obstackle that we have only a real-analytic calculus while α can only be expected to be smooth. A detailed discussion will be given elsewhere.

1.12.4. *Outlook.* The classical, non-quantized version of the canonical (anti-) commutation relations, the canonical Poisson brackets

$$\{\Xi_i(x), \Sigma_j(y)\} = \delta_{ij}\delta(x-y) \tag{1.12.3}$$

where $\Sigma_j := \big(\partial L/\partial(\partial_0 \Xi_j)\big)[\Xi]$ are the canonical momenta, suggests the introduction of the two form

$$\omega^{\text{Cau}} = \sum_{i=1}^{N^\Phi+N^\Psi} \int_{\mathbb{R}^3} dx \delta\Xi_i(x)\delta\Sigma_i(x)$$

on the smf of Cauchy data M^{Cau}; here δ is the exterior derivative for forms on the smf M^{Cau} (cf. [42] for a detailed theory), and the product under the integral is the exterior product of one forms.

This equips the smf M^{Cau} with a symplectic structure, and (1.12.3) holds in "smeared" form:

$$\Big\{\int dx f(x)\Xi_i(x), \int dx g(x)\Sigma_j(x)\Big\} = \int dx f(x) g(x)$$

for $f, g \in \mathcal{D}(\mathbb{R}^3)$ (all integrals over $\mathbb{R}^3$). The necessity of the buffer functions f, g is connected with the fact that on an infinite-dimensional symplectic (super-)manifold, the Poisson bracket is defined only on a subalgebra of the (super-)function algebra. Thus, the non-smeared writing (1.12.3) used in the textbooks of physicists is highly symbolic since the r.h.s. is not a well-defined superfunction.

With the aid of the isomorphism $M^{\text{Cau}} \xrightarrow{\Xi^{\text{sol}}} M^{\text{sol}}$, one can carry over ω^{Cau} to a symplectic structure ω^{sol} on M^{sol}; we will show elsewhere that this is Lorentz invariant, and hence an intrinsic structure (cf. also [47] for an alternative construction. This paper constructs heuristically a pseudo-Kähler structure on M^{sol}; however, the well-definedness of the latter is not clear).

This symplectic structure makes it possible to rewrite the field equations in Hamiltonian form. In view of this, it is natural to call the smf M^{sol} also the *covariant phase space* of the theory (cf. [18, 17.1.2], [11]).

The symplectic smf M^{Cau} might be the starting point for a geometric quantization. Of course, it is a rather tricky question what the infinite-dimensional substitute for the symplectic volume needed for integration should be; we guess that it is some improved variant of Berezin's functional integral (cf. [3]). Note, however, that although the integration domain M^{sol} is isomorphic to the linear

supermanifold M^{Cau}, this isomorphism is for a model with interaction highly Lorentz-non-invariant, and Berezin's functional integral makes use of that linear structure.

Also, in the interacting case, the usual problems of quantum field theory, in particular renormalization, will have to show up on this way, too, and the chances to construct a Wightman theory are almost vanishing. Nevertheless, it might be possible to catch some features of the physicist's computational methods (in particular, Feynman diagrams), overcoming the present mathematician's attitude of contempt and disgust to these methods, and giving them a mathematical description of Bourbakistic rigour.

What certainly can be done is a mathematical derivation of the rules which lead to the tree approximation S_{tree} of the scattering operator. S_{tree} should be at least a well-defined power series; of course, the wishful result is that in a theory without bounded states, S_{tree} is defined as an automorphism of the solution smf M^{free} of the free theory.

1.13. **Example: Fermions on a lattice.** As an interesting example for the description of configuration spaces as supermanifolds, we re-describe the standard framework of Euclidian Yang-Mills-Higgs-Dirac theory on a finite lattice (cf. [48]) in supergeometric terms. Here the finite-dimensional smf calculus is still sufficient.

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice. Let $L^* := \{(x,y) \in \Lambda^2 : \ \|x-y\| = 1\}$ the set of its links, and let $L := \{(x,y) \in L^* : \ \forall i \ \ y_i \geq x_i\}$.

Let G be a compact Lie group; we recall that G has a canonical real-analytic structure.

A *configuration of the gauge field* is a map $g : L^* \to G, \ \ (x,y) \mapsto g_{xy}$ with $g_{xy} g_{yx} = 1$; thus, the configuration space is the real-analytic manifold G^L. The *Wilson-Polyakov action* is the real-analytic function

$$S_{\mathrm{YMW}} : G^L \to \mathbb{R}, \quad g \mapsto -\frac{1}{2} \sum \chi(g_{vw} g_{wx} g_{xy} g_{yu})$$

where the sum runs over all plaquettes of Λ, i. e. all tuples $(x,y,u,v) \in G^4$ for which $(x,y), (y,u) \in L$, $(u,v), (v,w) \in L^*$; tuples which differ only by cyclic permutations are identified. Also, $\chi : G \to \mathbb{R}$ is some character of G belonging to a locally exact representation. (We ignore all wave function renormalization and coupling constants.)

For the *Higgs field*, we need a finitedimensional real Hilbert space $\mathcal{V}_H$ and an orthogonal representation $U_{\mathrm{H}} : G \to \mathrm{O}(\mathcal{V}_{\mathrm{H}})$. A *configuration of the Higgs field* is a map $\phi : \Lambda \to \mathcal{V}_{\mathrm{H}}$; thus, the configuration space for the Higgs field is the vector space ${\mathcal{V}_{\mathrm{H}}}^{\Lambda}$. The Higgs action is the real-analytic function

$$M_{\mathrm{bos}} = G^L \times {\mathcal{V}_{\mathrm{H}}}^{\Lambda} \xrightarrow{S_{\mathrm{H}}} \mathbb{R}, \quad (g, \phi) \mapsto -\frac{1}{2} \sum_{(x,y) \in L} (\phi(x), U_{\mathrm{H}}(g_{xy}) \phi(y)) + \sum_{x \in \Lambda} V(|\phi(x)|)$$

where V is a fixed polynomial of even degree ≥ 4 with positive highest coefficient.

Turning to the Dirac field, we assume to be given a Clifford module $\mathcal{V}_{\mathrm{S}}$ ("spinor space" in [48]) over $\mathrm{Cliff}(\mathbb{R}^{d+1})$; that is, $\mathcal{V}_{\mathrm{S}}$ is a finitedimensional complex Hilbert space together with selfadjoint operators $\gamma_i \in \mathrm{End}_{\mathbb{C}}(\mathcal{V}_{\mathrm{S}}) \ \ (i = 0, \ldots, d)$ such that $\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}$.

Also, we need a finitedimensional real Hilbert space $\mathcal{V}_{\mathrm{G}}$ ("gauge" or "colour" space) and an orthogonal representation $U_{\mathrm{G}} : G \to \mathrm{O}(\mathcal{V}_{\mathrm{G}})$. Let $\mathcal{V}_{\mathrm{F}} := \mathcal{V}_{\mathrm{S}} \otimes_{\mathbb{R}} \mathcal{V}_{\mathrm{G}}$, which is a complex Hilbert space, and consider it purely odd. Let (e_α), (f_a) be orthonormal bases of $\mathcal{V}_{\mathrm{S}}$ and $\mathcal{V}_{\mathrm{G}}$, respectively; thus, the elements

$$\Psi_{\alpha a} := e_\alpha \otimes f_a$$

form an orthonormal basis of $\mathcal{V}_{\mathrm{F}}$. Now take one copy $\mathcal{V}_{\mathrm{F}}(x)$ of $\mathcal{V}_{\mathrm{F}}$ for each lattice site $x \in \Lambda$, with basis $(\Psi_{\alpha a}(x))$, and let $\overline{\mathcal{V}_{\mathrm{F}}(x)}$ be the "exterior" conjugate of $\mathcal{V}_{\mathrm{F}}(x)$, i.e. the Hilbert space consisting of all elements $\overline{v}$ with $v \in \mathcal{V}_{\mathrm{F}}(x)$; Hilbert space structure and G-action are fixed by requiring that the map $\mathcal{V}_{\mathrm{F}}(x) \to \overline{\mathcal{V}_{\mathrm{F}}(x)}, \quad v \mapsto \overline{v}$, be antilinear, norm-preserving, and G-invariant.

Setting

$$\mathcal{V} := \bigoplus_{x\in\Lambda} \mathcal{V}_{\mathrm{F}}(x), \quad \overline{\mathcal{V}} := \bigoplus_{x\in\Lambda} \overline{\mathcal{V}_{\mathrm{F}}(x)}$$

(of course, the "$\otimes$" of [48] is a misprint), the space $\mathcal{V}\oplus\overline{\mathcal{V}}$ has a natural hermitian structure (conjugation acts as the notation suggests). Let $\mathcal{V}_r := \{v + \overline{v};\ v \in \mathcal{V}\}$ be its real part; thus, $\mathcal{V}$ identifies with the complexification of $\mathcal{V}_r$. Now the Grassmann algebra

$$\mathcal{G}_\Lambda := \bigwedge(\mathcal{V} \oplus \overline{\mathcal{V}}),$$

is nothing but the algebra of superfunctions on the hermitian supermanifold $M_{\mathrm{ferm}} := \mathrm{L}(\mathcal{V}_r)$:

$$\mathcal{G}_\Lambda = \mathcal{O}(M_{\mathrm{ferm}})$$

(recall that $\mathcal{V}_r$, being a real Hilbert space, identifies canonically with its dual), thus, M_{ferm} is the configuration supermanifold for the fermionic field strengthes. The field strengthes now appear as odd superfunctions:

$$\Psi_{\alpha a}(x), \overline{\Psi_{\alpha a}(x)} \in \mathcal{O}(M_{\mathrm{ferm}})_{\mathbf{1}};$$

in the language of [39], the $\Psi_{\alpha a}(x)$ form a chiral coordinate system on M_{ferm}.

The configuration space for the whole system is the hermitian supermanifold

$$M = M_{\mathrm{bos}} \times M_{\mathrm{ferm}} = G^L \times {\mathcal{V}_{\mathrm{H}}}^\Lambda \times \mathrm{L}(\mathcal{V}_r).$$

The algebra of global superfunctions is the Grassmann algebra generated by $\Psi_{\alpha a}(x), \overline{\Psi_{\alpha a}}(x)$ with the coefficients being real-analytic functions of the g_{xy}, $\phi(x)$.

(Actually, [48] uses instead of this the Grassmann algebra $\mathcal{A}_\Lambda$ with the same generators but with the coefficients being continuous bounded functions of the g_{xy}, $\phi(x)$; but that difference does not really matter.)

Turning to the fermionic action, fix parameters $r \in (0, 1]$, $\theta \in [0, \pi/2)$; for their meaning, we refer to the literature. For $(x, y) \in L$ set

$$\Gamma^{xy}_{\alpha\beta} := r \exp(\mathrm{i}\,\theta\gamma_d) + \sum_{j=0}^{d} (x_j - y_j)(\gamma_j)_{\alpha\beta};$$

since x, y are neighbours, only one term of the sum is non-zero. Also, observe that $g \mapsto U_{\mathrm{G}}(g_{xy})_{ab}$ is a real-analytic function on G^L. Therefore we can form the superfunction

$$S_{\mathrm{F}} := \frac{1}{2} \sum_{(x,y)\in L} \sum_{a,b,\alpha,\beta} \overline{\Psi_{\alpha a}(x)}\Gamma^{xy}_{\alpha\beta} U_{\mathrm{G}}(g_{xy})_{ab} \Psi_{\beta b}(y) + \frac{1}{2} \sum_{x\in\Lambda} \sum_{a,\alpha,\beta} \overline{\Psi_{\alpha a}(x)} \Big(M - rd \exp(\mathrm{i}\,\theta\gamma_d)_{\alpha\beta}\Big) \Psi_{\beta a}(x),$$

which is the action for the fermion field. Here M is the mass of the Dirac field.

The total action is now a superfunction on M:

$$S_{\mathrm{tot}} := S_{\mathrm{YMW}} + S_{\mathrm{H}} + S_{\mathrm{F}} \in \mathcal{O}(M)_{\mathbf{0}}.$$

We look at the action of the group of gauge transformations G^Λ, which is the Lie group of all maps $h : \Lambda \to G$: By

$$(h \cdot g)_{xy} := h_x g_{xy} {h_y}^{-1}, \quad (h \cdot \phi)(x) := U_{\mathrm{H}}(h_x)\phi(x),$$

we get an action

$$G^\Lambda \times M_{\mathrm{bos}} \to M_{\mathrm{bos}} \tag{1.13.1}$$

of the group of gauge transformations on the configuration space for the gauge and Higgs fields. On the other hand, we have a representation of $U_{\mathrm{F}} : G^\Lambda \to \mathrm{O}(\mathcal{V}_r)$ by restricting the natural unitary

action of G^Λ on $\mathcal{V}$ to the real part (this works due to the reality of the representation U_{G}). We get a representation (cf. [42, 5.18-20] for some basic notions)

$$a : G^\Lambda \times M_{\mathrm{ferm}} \to M_{\mathrm{ferm}}, \quad a^*(\Psi_{\alpha a}(x)) = U_{\mathrm{G}}(h_x)\Psi_{\alpha a}(x) \tag{1.13.2}$$

of G^Λ on the linear supermanifold M_{ferm}. (1.13.1), (1.13.2) together yield an action

$$a : G^\Lambda \times M \to M$$

of the group of gauge transformations on the configuration supermanifold, and S is now an invariant function, i.e. $a^*(S) = {\mathrm{pr}_2}^*(S)$.

Let $\mathcal{O}_b(M_{\mathrm{bos}})$ be the subspace of all $f \in \mathcal{O}(M_{\mathrm{bos}})$ which grow only polynomially in Higgs direction, i. e. there exist $C, p > 0$ such that

$$|f(g,\phi)(x)| \le C(1 + |\phi(x)|^p)$$

for $x \in \Lambda$, and let

$$\mathcal{O}_b(M) := \mathcal{O}_b(M_{\mathrm{bos}}) \otimes \mathcal{O}(M_{\mathrm{ferm}}) \subseteq \mathcal{O}(M).$$

Now we can interpret the *mean value* as a Berezin integral: For $P \in \mathcal{O}_b(M)$, we may form

$$\langle P \rangle_\Lambda := \int dg \prod_{k,x} d\phi_k(x) \int \prod_{\alpha,a,x} d\Psi_{\alpha a}(x) d\overline{\Psi_{\alpha a}(x)} \;\; P \exp(-S_{\mathrm{tot}}/\hbar). \tag{1.13.3}$$

Here the inner integral is Berezin integration along the fibres of the super vector bundle $M = M_{\mathrm{bos}} \times \mathrm{L}(\mathcal{V}_r) \xrightarrow{\mathrm{pr}} M_{\mathrm{bos}}$, producing an ordinary function on M_{bos}.

The $\phi_k(x)$ $(k = 1, \dots, \dim \mathcal{V}_{\mathrm{H}})$ are orthonormal coodinates on the x-component of ${\mathcal{V}_{\mathrm{H}}}^\Lambda$; thanks to the exponential factor and our growth condition, the integral over them is finite. Finally, dg is the normalized Haar measure on G^L. The sign ambiguity arising from the missing order of the $\phi_k(x)$ is resolved by fixing $\langle 1 \rangle_\Lambda > 0$.

Now we suppose the existence of *time reflection* as a fixpoint-free involutive map $r : \Lambda \to \Lambda$ which respects the link structure, $(r \times r)(L^*) \subseteq L^*$. By permuting the factors, r yields an involution $r : M_{\mathrm{bos}} \to M_{\mathrm{bos}}$. Also, we get a morphism

$$r : M_{\mathrm{ferm}} \to M_{\mathrm{ferm}}, \quad r^*(\Psi_{\alpha a}(x)) = \sum_\beta \Psi_{\beta a}(rx)(\gamma_0)_{\alpha\beta}.$$

Altogether, we get an involutive morphism $r : M \to M$ and a new hermitian law

$$\theta(P) := \overline{r^*(P)}.$$

(Actually, the notation is unlucky in view of 1.5, because it hides the skew-linearity.) We also need a decomposition of the lattice

$$\Lambda = \Lambda_+ \cup \Lambda_-, \quad \Lambda_+ \cap \Lambda_- = \emptyset$$

such that $r(\Lambda_\pm) \subseteq \Lambda_\mp$ (cf. [48] for the background). We get a projection morphism $M = M_\Lambda \to M_{\Lambda_+}$, and hence an embedding $\mathcal{O}_b(M_{\Lambda_+}) \subseteq \mathcal{O}_b(M)$. *Osterwalder-Schrader positivity* now states that

$$\langle P\theta(P) \rangle \ge 0$$

for all $p \in \mathcal{O}_b(M_{\Lambda_+})$. Also, the scalar product

$$\mathcal{O}_b(M_{\Lambda_+}) \times \mathcal{O}_b(M_{\Lambda_+}) \to \mathbb{C}, \quad (P,Q) \mapsto \langle P\theta(Q) \rangle,$$

satisfies

$$\langle Q\theta(P) \rangle = \overline{\langle P\theta(Q) \rangle}; \tag{1.13.4}$$

hence, it equips $\mathcal{O}_b(M_{\Lambda_+})/\{P : \langle P\theta(P) \rangle = 0\}$ with the structure of a pre-Hilbert space the completion $\mathcal{H}$ of which is the *Euclidian state space*. (Thus, $\mathcal{O}_b(M_{\Lambda_+})$ plays here the same role as the space of polarized sections in geometric quantization.)

Remarks . (1) Note that working with a conventional, non-hermitian calculus would bring trouble here since (1.13.4) would acquire an additional factor $(-1)^{|Q||P|}$, and hence $\langle P\theta(P)\rangle$ would be for odd P not positive but imaginary.

(2) It would be interesting to know something about the "supermanifold of classical solutions" of the action S. That is, we may form the ideal sheaf $\mathcal{J}(\cdot) \subseteq \mathcal{O}_M(\cdot)$ generated by the "field equations"

$$\frac{\partial}{\partial g^i_{xy}}S, \quad \frac{\partial}{\partial \phi_k(x)}S, \quad \frac{\partial}{\partial \Psi_{\alpha a}(x)}S, \quad \frac{\partial}{\partial \overline{\Psi_{\alpha a}(x)}}S \ \in \mathcal{O}(M),$$

(here g^i_{xy} are local coodinates on the (x, y)-component of G^L) and take the factor space $M^{\mathrm{sol}} := (\operatorname{supp} \mathcal{O}/\mathcal{J}, \ \ \mathcal{O}/\mathcal{J})$. This is at least a real superanalytic space which, however, might have singularities. Note that, in contrast to the non-super situation, where the singular locus of an analytic space has codimension ≥ 1, it may on a superanalytic space be the whole space; it would be nice to show that this does not happen here.

Also, one should prove that in the classical limit $\hbar \to 0$, the integral (1.13.3) becomes asymptotically equal to an integral over M^{sol} (or its non-singular part). Indeed, in the bosonic sector, the exponential factor makes the measure accumulate on the subspace $\widetilde{M^{\mathrm{sol}}}$ of the minima of the action; however, in the fermionic sector, the picture is less clear.

2. Infinite-dimensional supermanifolds

An attentive reader of [44] will have noted that most of its material (with the exception of local functionals, differential polynomials a.s.o.) does not really depend on the fact that E is a function space. In fact, the theory can be developed in an abstract context, and one really should do so in order to get conceptual clarity which may be useful if a concrete situation does not fit into our framework. In that way, we will also establish the connection with usual, finite-dimensional supergeometry a la Berezin.

Some work has been done on infinite-dimensional supergeometry and its application onto classical fields in quantum field theory. Apart from the implicite appearance of infinite-dimensional supermanifolds in [3] (cf. [43]), the first work on the mathematical side is Molotkov [29]; however, his approach is not well suited for physical purposes. Cf. below for a discussion.

[1] uses an ad-hoc definition of smooth Banach smfs in order to describe mathematically the fermionic Faddeev-Popov ghost fields used by physicists for quantization of the Yang-Mills field.

In [24], infinite-dimensional supergeometry makes an implicite appearance in a more general approach to the quantization of systems with first-class constraints (cf. also the comment in [16]).

[34] and [16] use, on a physical level of rigour, ad-hoc generalizations of the Berezin-Kostant supergeometry framework for applying geometric quantization in field theory, in particular to fermionic fields.

Finally, the present author constructed in [40], [41], [42], the predecessors of the present paper, a rather general theory of complex- and real-analytic supermanifolds modelled over locally convex spaces.

Here, we will give an alternative description of this theory, using a traditional treatment via ringed spaces and charts. Also, we will treat only real-analytic supermanifolds with complete model spaces.

We will give only a short account on the abstract variant since the details should be clear from the material of [44].

2.1. **Formal power series.** Let E, F be complete $\mathbb{Z}_2$-lcs, and define the space $\mathcal{P}^{k|l}(E;F)$ of *F-valued $k|l$-forms on E* as the space of all $(k+l)$-multilinear continuous maps

$$u_{(k|l)} : \prod^{k} E_{\mathbf{0}} \times \prod^{l} E_{\mathbf{1}} \to F_{\mathbb{C}}$$

which satisfy the symmetry requirement

$$u_{(k|l)}(e_{\sigma(1)}, \dots, e_{\sigma(k)}, e'_{\pi(1)}, \dots, e'_{\pi(l)}) = \operatorname{sign}(\pi) u_{(k|l)}(e_1, \dots, e_k, e'_1, \dots, e'_l)$$

for all permutations σ, π. $\mathcal{P}^{k|l}(E;F)$ is a $\mathbb{Z}_2$-graded vector space; note that we do not distinguish a topology on it.

The space of *F-valued formal power series* on E is defined by

$$\mathcal{P}_f(E;F) := \prod_{k,l\geq 0} \mathcal{P}^{k|l}(E;F);$$

thus, its elements are formal sums $u = \sum_{k,l\geq 0} u_{(k|l)}$ where $u_{(k|l)} \in \mathcal{P}^{k|l}(E;F)$. The rule

$$\overline{u}_{(k|l)}(e_1, \dots, e_k, e'_1, \dots, e'_l) := \overline{u_{(k|l)}(e_1, \dots, e_k, e'_l, e'_{l-1}, \dots, f_1)}$$

$$= (-1)^{\binom{l}{2}} \overline{u_{(k|l)}(e_1, \dots, e_k, e'_1, \dots, e'_l)}$$

turns $\mathcal{P}_f(E;F)$ into a hermitian vector space. The *product*

$$\mathcal{P}^{k|l}(E;F) \times \mathcal{P}^{k'|l'}(E;F') \to \mathcal{P}^{k+k'|l+l'}(E; F\widehat{\otimes}F')$$

is defined by

$$(u \otimes v)_{(k+k'|l+l')}(e_1, \dots, e_{k+k'}, e'_1, \dots, e'_{l+l'}) =$$
$$= \sum (\pm) \binom{k+k'}{k}^{-1} \binom{l+l'}{l}^{-1} u_{(k|l)}(e_{p_1}, \dots, e_{p_k}, e'_{q_1}, \dots, e'_{q_l}) \otimes v_{(k'|l')}(e_{p'_1}, \dots, e_{p'_k}, e'_{q'_1}, \dots, e'_{q'_l}).$$

(the sign "$\otimes$" on the l.h.s. is somewhat abusive). Here the sum runs over all $\binom{k+k'}{k}\binom{l+l'}{l}$ partitions

$$\{1, \dots, k+k'\} = \{p_1, \dots, p_k\} \sqcup \{p'_1, \dots, p'_{k'}\}, \quad p_1 \leq \dots \leq p_k, \quad p'_1 \leq \dots \leq p'_{k'}$$

$$\{1, \dots, l+l'\} = \{q_1, \dots, q_l\} \sqcup \{q'_1, \dots, q'_{l'}\}, \quad q_1 \leq \dots \leq q_l, \quad q'_1 \leq \dots \leq q'_{l'}$$

and $(\pm)$ is given by the sign rule:

$$(\pm) := (-1)^{|v|(|q_1|+\dots+|q_l|)} \operatorname{sign}(\pi)$$

where π is the permutation $(q_1, \dots, q_l, q'_1, \dots, q'_{l'})$ of $\{1, \dots, l+l'\}$ (cf. [41, Prop. 2.3.1]).

The product turns $\mathcal{P}_f(E;\mathbb{R})$ into a $\mathbb{Z}_2$-commutative hermitian algebra and each $\mathcal{P}_f(E;F)$ into a hermitian module over that algebra. In both situations, we will usually write simply uv instead of $u \otimes v$.

Remark . An F-valued formal power series in the sense of [44, 2.3],

$$K[\Phi|\Psi] = \sum_{k,l\geq 0} \frac{1}{k!l!} \sum_{I|J} \int dX dY K^{I|J}(X|Y) \prod_{m=1}^{k} \Phi_{i_m}(x_m) \cdot \prod_{n=1}^{l} \Psi_{j_n}(y_n) \ \in \mathcal{P}(F),$$

defines an F-valued formal power series $K \in \mathcal{P}(D;F)$ in the sense above, $K = \sum_{k,l} K_{(k|l)}$ with

$$\text{(2.1.1)} \quad K_{(k|l)} : \prod^{k} D_{\mathbf{0}} \times \prod^{l} D_{\mathbf{1}} \to F_{\mathbb{C}}, \qquad (\phi^1, \dots, \phi^k, \psi^1, \dots, \psi_l) \mapsto$$
$$\frac{1}{k!l!} (-1)^{l(l-1)/2} \int dX dY \sum_{I|J} K^{I|J}(X|Y) \prod_{m=1}^{k} \phi^m_{i_m}(x_m) \cdot \prod_{n=1}^{l} \Pi\psi^n_{j_n}(y_n)$$

where $D = \mathcal{D}(\mathbb{R}^d) \otimes V$, and $V = V_{\mathbf{0}} \oplus V_{\mathbf{1}}$ is the field target space. (The apparently strange parity shift Π was motivated by the wish to have Ψ as an odd symbol.)

Moreover, if E is an admissible function space in the sense of [44, 3.1], i. e. E is a $\mathbb{Z}_2$-graded complete locally convex space with continuous inclusions $D \subseteq E \subseteq \mathcal{D}'(\mathbb{R}^d) \otimes V$, then we have $K \in \mathcal{P}_f(E; F)$ iff for all $k, l \geq 0$, (2.1.1) extends to a continuous map $\bigotimes^k E_{\mathbf{0}} \otimes \bigotimes^l E_{\mathbf{1}} \to F_{\mathbb{C}}$. In that way, we get a natural identification between $\mathcal{P}_f(E; F)$ as defined in [44, 3.1] and the $\mathcal{P}_f(E; F)$ defined here.

(In [44, 3.1], we had also assigned to $K_{(k|l)}[\Phi|\Psi]$ the linear map

$$\bigotimes^k D_{\mathbf{0}} \otimes \bigotimes^l \Pi D_{\mathbf{1}} \to F_{\mathbb{C}}, \qquad \bigotimes_{m=1}^k \phi^m \otimes \bigotimes_{n=1}^l \Pi\psi^n \mapsto (-1)^{l(l-1)/2} \cdot (\text{second line of (2.1.1)}).$$

Note, however, that the parity of this map differs from that of $K[\Phi|\Psi]$ by the parity of l.)

2.2. **Analytic power series.** Let be given continuous seminorms $p \in \mathrm{CS}(E)$, $q \in \mathrm{CS}(F)$. We say that $u \in \mathcal{P}_f(E; F)$ *satisfies the* (q, p)*-estimate* iff we have for all k, l with $k + l > 0$ and all $e_1, \dots, e_k \in E_{\mathbf{0}}$, $e'_1, \dots, e'_l \in E_{\mathbf{1}}$ the estimate

$$q(u_{(k|l)}(e_1, \dots, e_k, e'_1, \dots, e'_l)) \leq p(e_1) \cdots p(e_k) p(e'_1) \cdots p(e'_l)$$

(we extend every $q \in \mathrm{CS}(F)$ onto $F_{\mathbb{C}}$ by $q(f + \mathrm{i}\, f') := q(f) + q(f')$). We call u *analytic* iff for each $q \in \mathrm{CS}(F)$ there exists a $p \in \mathrm{CS}(E)$ such that u satisfies the (q, p)-estimate.

Now every $k|l$-form $u_{(k|l)} \in \mathcal{P}^{k|l}(E; F)$ is analytic, due to its continuity property, and analyticity of a formal power series is just a joint-continuity requirement onto its coefficients $u_{(k|l)}$.

The analytic power series form a hermitian subspace $\mathcal{P}(E; F)$ of $\mathcal{P}_f(E; F)$. Moreover, tensor product of analytic power series and composition with linear maps in the target space produce analytic power series again.

For $e" \in E$, the *directional derivative* $\partial_{e"}$ is defined by

$$(\partial_{e"} u)_{(k|l)}(e_1, \dots, e_k, e'_1, \dots, e'_l) := \begin{cases} (k+1) u_{(k+1|l)}(e", e_1, \dots, e_k, e'_1, \dots, e'_l) & \text{for } |e"| = 0 \\ (-1)^{|u|}(l+1) u_{(k|l+1)}(e_1, \dots, e_k, e"_1, e'_1, \dots, e'_l) & \text{for } |e"| = 1. \end{cases}$$

$\partial_{e"}$ maps both $\mathcal{P}_f(E; F)$ and $\mathcal{P}(E; F)$ into themselves, and it acts as derivation on products:

$$\partial_e(u \otimes v) = \partial_e u \otimes v + (-1)^{|e||u|} u \otimes \partial_e v.$$

The abstract analogon of the functional derivative of K is the linear map

$$E \to \mathcal{P}_f(E; F), \qquad e \mapsto \partial_e K.$$

Suppose that F is a $\mathbb{Z}_2$-graded Banach space, and fix $p \in \mathrm{CS}(E)$. Set

$$\|u_{(k|l)}\|_p := \inf\{c > 0 : \quad u_{(k|l)} \text{ satisfies a } (c^{-1}\|\cdot\|, p)\text{-estimate}\}$$

for $k + l > 0$, and define $\|\cdot\|$ on $0|0$-forms to be the norm in $F_{\mathbb{C}}$. Then

$$\mathcal{P}(E, p; F) = \{u \in \mathcal{P}(E; F) : \quad \|u\|_p := \sum_{k,l \geq 0} \|u_{(k|l)}\| < \infty\}$$

is a Banach space. Moreover, for any E we have

$$\mathcal{P}(E; F) = \bigcup_{p \in \mathrm{CS}(E)} \mathcal{P}(E, p; F).$$

Now if F' is another $\mathbb{Z}_2$-graded Banach space then

$$\|u \otimes v\|_p \leq \|u\|_p \|v\|_p$$

for $u \in \mathcal{P}(E; F)$, $v \in \mathcal{P}(E; F')$. In particular, $\mathcal{P}(E, p; \mathbb{R})$ is a Banach algebra.

Note that for every $p \in \mathrm{CS}(E)$, $c > 1$, the directional derivative ∂_e maps $\mathcal{P}(E,p;F) \to \mathcal{P}(E,cp;F)$ for any $e \in E$.

Remarks . (1) A $k|l$-form $u_{(k|l)}$ lies in $\mathcal{P}(E,p;F)$ iff it factors through $\hat{E}_p$; in that case, $\|u_{(k|l)}\|$ is just its supremum on the $k+l$-fold power of the unit ball of this space.

(2) Remark 3.2.2 of [44] carries over, linking the approach here with [41]: Fixing $u \in \mathcal{P}_f(E;F)$ and $k, l \geq 0$ we get a continuous map

$$\mathrm{S}^k\, E_{\mathbf{0},\mathbb{C}} \cdot \mathrm{S}^l\, E_{\mathbf{1},\mathbb{C}} \to F_{\mathbb{C}}, \qquad e_1 \cdots e_k e_1' \cdots e_l' \mapsto k!l!u(e_1,\ldots,e_k,e_1',\ldots,e_l')$$

(using notations of [41]; the topology on the l. h. s. is induced from the embedding into $\mathrm{S}\, E_{\mathbb{C}}$). Using Remark 2.1.(2) of [41] we get a bijection

$$\mathcal{P}_f(E;F) \to \prod_{k\geq 0} \mathcal{L}(\mathrm{S}^k\, E_{\mathbb{C}}, F_{\mathbb{C}})$$

(symmetric algebra in the super sense). The r. h. s. is somewhat bigger than $\mathcal{L}(\mathrm{S}\, E_{\mathbb{C}}, F_{\mathbb{C}}) = \mathrm{P}(E;F)$, due to the absence of growth conditions. Having identified both sides, one shows for $u \in \mathcal{P}_f(E;F)$ the estimates

$$\|u\|_{U_{p/2}} \leq \|u\|_p \leq \|u\|_{U_{2p}}$$

(cf. [41, 2.5] for the notations $\|\cdot\|_U$, $\mathrm{P}(E,U;F)$), and hence

$$\mathrm{P}(E,U_{p/2};F) \subseteq \mathcal{P}(E,p;F) \subseteq \mathrm{P}(E,U_{2p};F), \qquad \mathcal{P}(E;F) = \mathrm{P}(E;F).$$

*

2.3. **Insertions.** On the level of formal power series, we will define $u[v] \in \mathcal{P}_f(E';F)$ with the data

$$u \in \mathcal{P}_f(E;F), \qquad v \in \mathcal{P}_f(E';E)_{\mathbf{0}}, \qquad v_{(0|0)} = 0.$$

We split $v = v_{(\mathbf{0})} + v_{(\mathbf{1})}$ with $v_{(\mathbf{i})} \in \mathcal{P}_f(E';E_{\mathbf{i}})_{\mathbf{0}}$ $(\mathbf{i} = \mathbf{0}, \mathbf{1})$, and we set

$$u[v] := \sum_{k,l\geq 0} \Big\langle u_{(k|l)}, \bigotimes^k v_{(\mathbf{0})} \otimes \bigotimes^l v_{(\mathbf{1})} \Big\rangle. \tag{2.3.1}$$

Here $\bigotimes^k v_{(\mathbf{0})} \otimes \bigotimes^l v_{(\mathbf{1})} \in \mathcal{P}_f(E'; \bigotimes^k E_{\mathbf{0}} \otimes \bigotimes^l E_{\mathbf{1}})$ is the product, and $u_{(k|l)}$ is viewed as continuous linear map $\bigotimes^k E_{\mathbf{0}} \otimes \bigotimes^l E_{\mathbf{1}} \to F_{\mathbb{C}}$,

$$\big\langle u_{(k|l)}, e_1 \otimes \cdots \otimes e_k \otimes e_1' \otimes \cdots \otimes e_l' \big\rangle := u_{(k|l)}(e_1,\ldots,e_k,e_1',\ldots,e_l').$$

Thus, each term of (2.3.1) makes sense as element of $\mathcal{P}_f(E';F)$, and by the same arguments as in [44, 3.3], (2.3.1) is a well-defined formal power series.

If we want to lift the condition $v_{(0|0)} = 0$ we have to introduce more hypotheses. Suppose that

(1) $v \in \mathcal{P}(E';E)_{\mathbf{0}}$, $u \in \mathcal{P}(E,p;F)$ where F is a Banach space and $p \in \mathrm{CS}(E)$, and

(2) there exists some $q \in \mathrm{CS}(E')$ with

$$i_p v \in \mathcal{P}(E',q;\hat{E}_p), \tag{2.3.2}$$

$$\|i_p v\|_q < 1. \tag{2.3.3}$$

Defining $u_{(k|l)}[v] \in \mathcal{P}(E';F)$ as above, one has $\|u_{(k|l)}[v]\| \leq \|u_k[v]\|_p \cdot \|i_p v_{(\mathbf{0})}\|_q^k \cdot \|i_p v_{(\mathbf{1})}\|_q^l$, and therefore the series (2.3.1) converges in $\mathcal{P}(E',q;F)$.

Thus, for varying u, we get a linear map of Banach spaces

$$\mathcal{P}(E,p;F) \to \mathcal{P}(E',q;F), \quad u \mapsto u[v],$$

of norm $\leq 1/(1 - \|i_p v\|_q)$; for $F = \mathbb{R}$, this is a homomorphism of Banach algebras.

More generally, let again the data (1) be given, and replace (2) by the weaker condition

$$p(v_{(0|0)}) < 1.$$

Then $u[v] \in \mathcal{P}(E'; F)$ still makes sense: We can choose $q \in \mathrm{CS}(E')$ with (2.3.2). Now, replacing q by cq with a suitable $c > 1$, we can achieve (2.3.3).

Using these results it is easy to show that the insertion $u[v]$ where u, v are analytic and $v_{(0|0)} = 0$ is analytic again.

Remark . The connection of the insertion mechanism with [41] is the following: If u, v are formal power series and $v_{(0|0)} = 0$ then $u[v]$ here is nothing but $u \circ \exp(v)$ of [41, 2.3]. More generally, if $u,\ v$ are analytic and $\phi := v_{(0|0)} \neq 0$ then $u[v] = (\mathrm{t}_\phi\, u) \circ \exp(v - \phi)$ where $\exp(v-\phi)$ is the cohomomorphism $\mathrm{S}\, E'_{\mathbb{C}} \to \mathrm{S}\, E_{\mathbb{C}}$ determined by $v - \phi$.

Whenever insertion is defined, it is associative, i. e. $u[v[w]]$ makes unambiguous sense if it is defined. (A direct proof is sufficiently tedious. Cf. the approach of [41] which saves that work.)

The power series $x = x_E \in \mathcal{P}(E; E)$ defined by

$$x_{(1|0)}(e_{\mathbf{0}}) = e_{\mathbf{0}}, \qquad x_{(0|1)}(e_{\mathbf{1}}) = e_{\mathbf{1}}$$

$(e_{\mathbf{i}} \in E_{\mathbf{i}})$, $x_{(k|l)} = 0$ for all other k, l, is the unit element under composition: for $u \in \mathcal{P}(E; F)$,

$$u[x_E] = x_F[u] = u.$$

x is the abstract analogon to the Ξ considered in [44, 3.3]. Sometimes, we call x the *standard coordinate*, and we will write power series in the form $u = u[x]$.

Suppose again that F is a Banach space, and let

$$u \in \mathcal{P}(E, p; F), \qquad e \in E_{\mathbf{0}}, \qquad c := 1 - p(e) > 0.$$

We define the *translation* $\mathrm{t}_e\, u \in \mathcal{P}(E, cp; F) \subseteq \mathcal{P}(E; F)$ by

$$\mathrm{t}_e\, u[x] := u[x + e].$$

Here e is viewed as constant power series $e \in \mathcal{P}(E; E_{\mathbf{0}})$. We have the *Taylor formula*

$$\mathrm{t}_e\, u = \sum_{k \geq 0} \frac{1}{k!} \partial_e^k u;$$

the sum absolutely converges in $\mathcal{P}(E, cp; F)$. Thus

$$(\mathrm{t}_e\, u)_{(k|l)}(e_1, \ldots, e_k, e'_1, \ldots, e'_l) = \sum_{k' \geq 0} \binom{k+k'}{k} u_{(k+k'|l)}(\underbrace{e, \ldots, e}_{k' \text{ times}}, e_1, \ldots, e_k, e'_1, \ldots, e'_l).$$

Sometimes, we will write $u \circ v$ instead of $u[v]$, and call this the *composition of u with v.*

2.4. **Superfunctions.** Let E, F be complete $\mathbb{Z}_2$-lcs and $U \subseteq E_{\mathbf{0}}$ open. The material of [44, 3.5] carries over verbally. Thus, an *F-valued (real-analytic) superfunction on U* is a map

$$u : U \to \mathcal{P}(E; F), \qquad e \mapsto u_e, \tag{2.4.1}$$

which satisfies the following condition: whenever u_e satisfies a (q, p)-estimate we have for all $e' \in U$ with $p(e' - e) < 1$

$$i_q u_{e'} = \mathrm{t}_{e'-e}\, i_q u_e.$$

We call u_e the *Taylor expansion* of u at e.

We get a sheaf $\mathcal{O}^F(\cdot)$ of hermitian vector spaces on E, and the product globalizes to a bilinear map

$$\mathcal{O}^F(U) \times \mathcal{O}^G(U) \to \mathcal{O}^{F \otimes G}(U), (u, v) \mapsto u \otimes v.$$

In particular, the sheaf $\mathcal{O}(\cdot) := \mathcal{O}^{\mathbb{R}}(\cdot)$ of *scalar superfunctions* is a sheaf of $\mathbb{Z}_2$-commutative hermitian algebras, and each $\mathcal{O}^F(\cdot)$ becomes a hermitian $\mathcal{O}$-module sheaf.

If the odd part of E is finitedimensional, we get for $U \subseteq E_{\mathbf{0}}$ a canonical isomorphism

$$\mathcal{O}_E(U) \cong \mathrm{An}(U, \mathbb{C}) \otimes \mathrm{S}\, E_{\mathbf{1},\mathbb{C}}{}^* \tag{2.4.2}$$

where $\mathrm{S}\, E_{\mathbf{1}}{}^*$ is the symmetric algebra over the complexified dual of the odd part of E in the supersense, i. e. the exterior algebra in the ordinary sense.

Also, the Propositions 3.5.1, 3.5.2 from [44] carry over:

Proposition 2.4.1. *(i) Suppose that $u \in \mathcal{O}^F(U)$, $\quad u_e = 0$ for some $e \in U$, and that U is connected. Then $u = 0$.*

(ii) Every Banach-valued analytic power series defines a "function element": Assume that F is Banach, and let be given an element $v \in \mathcal{P}(E, p; F)$, $e \in E_{\mathbf{0}}$. Setting

$$U := e + \{e' \in E_{\mathbf{0}} : \quad p(e') < 1\}$$

there exists a unique F-valued superfunctional $u \in \mathcal{O}^F(U)$ with $u_e = v$. Explicitly, it is given by

$$u_{e+e'} := \mathrm{t}_{e'}\, v \tag{2.4.3}$$

for $p(e') < 1$.

(iii) If $v \in \mathcal{P}_{\mathrm{pol}}(E; F)$ is an analytic polynomial where E, F may be arbitrary $\mathbb{Z}_2$-lcs there exists a unique global F-valued superfunctional $u \in \mathcal{O}^F(E_{\mathbf{0}})$ with $u_0 = v$. Explicitly, it is given by (2.4.3) *again.* □

We conclude with a useful criterion for a map (2.4.1) to be a superfunction.

Fixing a $\mathbb{Z}_2$-lcs F, we call a subset $V \subseteq F_{\mathbf{0}}^* \cup F_{\mathbf{1}}^*$ *strictly separating* iff there exists a defining system of seminorms $C \subseteq \mathrm{CS}(F)$ such that for all $p \in C$, $f \in F$ with $p(f) \neq 0$ there exists some $f^* \in V$ which satisfies $|f^*| \leq Kp$ with some $K > 0$ and $f^*(f) \neq 0$ (in other words, we require that for every $p \in C$ the set of that $f^* \in V$ which factorize through $\hat{F}_p$ separate that Banach space).

Obviously, every strictly separating V is separating (i. e. it separates the points of F). Conversely, if F is a $\mathbb{Z}_2$-Banach space then every separating $V \subseteq F^*$ is strictly separating. On the other hand, if F carries the weak topology $\sigma(F, F^*)$ then $V \subseteq F_{\mathbf{0}}^* \cup F_{\mathbf{1}}^*$ is strictly separating iff it generates F^* algebraically.

Proposition 2.4.2. *Let $U \subseteq E_{\mathbf{0}}$ be open, let F be a $\mathbb{Z}_2$-lcs, let $V \subseteq F^*$ be strictly separating, and let be given a map* (2.4.1) *such that for all $f^* \in V$, the map $e \mapsto \langle f^*, u_e \rangle \in \mathcal{P}(E; \mathbb{C})$ is an element of $\mathcal{O}(U)$. Then* (2.4.1) *is an element of $\mathcal{O}^F(U)$.*

Proof. First we treat the case that F is a Banach space. Fix $e \in E$ and choose $p \in \mathrm{CS}(E)$ with $u_e \in \mathcal{P}(E, p; F)$. For $e' \in E_{\mathbf{0}}$ with $p(e') < 1$, we get $\langle f^*, u_{e+e'} - t_{e'} u_e \rangle = 0$ for all $f^* \in V$; since V separates F we get $u_{e+e'} = t_{e'} u_e$, and the assertion follows.

Now let F be an arbitrary $\mathbb{Z}_2$-lcs. It follows from the Banach case and the definition of being strictly separating that for every $p \in C$ the assignment $e \mapsto i_p \circ u_e$ is an element of $\mathcal{O}^{\hat{F}_p}(U)$. The assertion follows. □

Surprisingly, the conclusion becomes false if V is only supposed to be separating. For a counterexample, cf. [42, 4.1].

2.5. **Superfunctions and ordinary functions.** Before proceeding, we recall from [44] the definition of real-analytic maps between locally convex spaces:

Definition 2.5.1. Let E, F be real locally convex vector spaces (no $\mathbb{Z}_2$grading), and let $U \subseteq E$ be open. A map $v : U \to F$ is called *real-analytic* iff there exists a family $(u_k)_{k\geq 0}$ of continuous maps

$$u_k : U \times \underbrace{E \times \cdots \times E}_{k \text{ times}} \to F$$

which are symmetric and multilinear in the last k arguments such that for each $q \in \mathrm{CS}(F)$, $e \in U$ there exists some open $V \ni 0$ with $e + V \subseteq U$ such that for all $e' \in V$

$$\sum_{k\geq 0} i_q(u_k(e, e', \dots, e')) = i_q v(e + e')$$

with absolute convergence in the Banach space $\hat{F}_q$, which is uniform in e', i. e.

$$\sup_{e'\in V} \sum_{k\geq k'} q(u_k(e, e', \dots, e')) \to 0$$

for $k' \to \infty$.

Of course, the u_k are the Gateaux derivatives of v.

We denote by $\mathrm{An}(U, F)$ the set of all real-analytic maps from U to F.

Remarks . (1) Of course, if F is Banach then it is sufficient to check the condition only for q being the original norm, i. e., i_q is the identity map. On the other hand, there exist (somewhat pathological) examples where it is impossible to find a common V for all $q \in \mathrm{CS}(F)$. Cf. [41] for a discussion and for the connection of real-analytic maps with holomorphic maps between complex locally convex spaces.

(2) The assignment $U \mapsto \mathrm{An}(U, F)$ is a sheaf of vector spaces on E. Also, the class of real-analytic maps is closed under pointwise addition and tensor product as well as under composition.

As in [44, 3.6], every $u \in \mathcal{O}^F(U)$ determines a real-analytic *underlying function*

$$\tilde{u} : U \to F_{\mathbb{C}}, \qquad e \mapsto \tilde{u}(e) := u_e[0].$$

In terms of the Taylor expansion $u_0 \in \mathcal{P}(E, p; F)$ at the origin, one has the explicit formula

$$\tilde{u}(e) = \sum_{k\geq 0} (u_0)_{(k|0)}(\underbrace{e, \dots, e}_{k \text{ factors}})$$

valid for $e \in E_{\mathbf{0}}$, $p(e) < 1$; if u is polynomial then this formula holds for all $e \in E_{\mathbf{0}}$.

[44, Thm. 3.6.2] carries over:

Theorem 2.5.2. *(i) The map*

$$\mathcal{O}^F(U) \to \mathrm{An}(U, F_{\mathbb{C}}), \qquad u \mapsto \tilde{u}, \tag{2.5.1}$$

is surjective, i. e. every real-analytic map is the underlying functional of some superfunctional. Moreover, (2.5.1) *turns products into pointwise tensor products, and it turns hermitian conjugation into conjugation within $F_{\mathbb{C}}$:*

$$(\widetilde{u \otimes v})(e) = \tilde{u}(e) \otimes \tilde{v}(e), \qquad (\tilde{\overline{u}})(e) = \overline{\tilde{u}(e)}.$$

Also, it turns even directional derivative into the Gateaux derivative: For $e' \in E_{\mathbf{0}}$ we have

$$\widetilde{\partial_{e'} u}(e) = D\tilde{u}(e, e').$$

(ii) If $E_{\mathbf{1}} = 0$ then (2.5.1) is bijective. Explicitly, for a given real-analytic map $v : U \to F_{\mathbb{C}}$ the corresponding superfunctional $u \in \mathcal{O}^F(U)$ is given by $(u_e)_{(k|l)} = 0$ for $l \neq 0$,

$$(u_e)_{(k|0)}(e_1, \ldots, e_k) = D^k v(e, e_1, \ldots, e_k)$$

where $D^k v$ is the k-th Gateaux derivative of v. □

We will denote by $\mathcal{M}^F(\cdot) := \mathcal{O}^F(\cdot)_{\mathbf{0},\mathbb{R}}$ the even, real part of $\mathcal{O}^F$.

The material of [44, 3.7] carries over:

Proposition 2.5.3. *Given $v \in \mathcal{M}_E^{E'}(U)$, $u \in \mathcal{O}_{E'}^F(U')$ with $\operatorname{Im} \tilde{v} \subseteq U'$, the map*

$$U \to \mathcal{P}(E; F), \quad e \mapsto (u \circ v)_e := u_{\tilde{v}(e)}[v_e - \tilde{v}(e)]$$

is an element $u \circ v = u[v] \in \mathcal{O}^F(U)$ called the composition of u with v. □

The standard coordinate, which was introduces in 2.3 only as power series, now globalizes by assertion (iii) of the Proposition to a superfunction $x = x_E \in \mathcal{M}^E(E_{\mathbf{0}})$ which we call again the *standard coordinate*.

As in [44, 3.5], it is often advisable to distinguish the "expansion parameter" x^e at a point notationally from the standard coordinate x.

2.6. **Superdomains.** Given a complete $\mathbb{Z}_2$-lcs, we can assign to it the hermitian ringed space (cf. 1.5)

$$\mathrm{L}(E) := (E_{\mathbf{0}}, \mathcal{O});$$

we call it the *linear supermanifold modelled over E*. The assignment $E \mapsto \mathrm{L}(E)$ is the superanalogon of the usual procedure of viewing a vector space as manifold.

Also, we call every open subspace $U = (\operatorname{space}(U), \mathcal{O}|_U)$ of it a *superdomain modelled over E*, and we write (symbolically) $U \subseteq \mathrm{L}(E)$.

A *morphism of superdomains*

$$\phi : (U, \mathcal{O}_U) \to (V, \mathcal{O}_V) \tag{2.6.1}$$

which are modelled over $\mathbb{Z}_2$-lcs E, F, respectively, is a morphism of hermitian ringed spaces which satisfies the following condition:

There exists an element $\hat{\phi} \in \mathcal{M}^F(U)$ such that

(1) the underlying map of ϕ is given by the underlying function $\tilde{\hat{\phi}} : U \to F$ of $\hat{\phi}$, and

(2) the superfunction pullback is given by composition with $\hat{\phi}$: For open $V' \subseteq V$, $\phi^* : \mathcal{O}_V(V') \to \mathcal{O}_U(\phi^{-1}(V'))$ maps

$$u \mapsto u \circ \hat{\phi}. \tag{2.6.2}$$

Using the standard coordinates $x_U \in \mathcal{M}^E(U)$, $x_V \in \mathcal{M}^F(V)$, this rewrites to

$$u[x_V] \mapsto u[\hat{\phi}[x_U]]. \tag{2.6.3}$$

We note that $\hat{\phi}$ is uniquely determined by ϕ since for all $f^* \in F^*$ we have $\left\langle f^*, \hat{\phi} \right\rangle = \phi^*(f^*) \in \mathcal{O}(U)$ (on the r.h.s., we view f^* as linear superfunction on V).

Remark 2.6.1. We note that in case that both E, F are finitedimensional, the additional requirement of the existence of $\hat{\phi}$ is automatically satisfied and hence redundant; that is, every morphism (2.6.1) of hermitian ringed spaces is a morphism of superdomains. Indeed, if f_i is a basis of F and f^i the left dual bases of F^* then it follows from the hermitian, real-analytic version of Cor. 1.3.2 that $\hat{\phi} = \sum \phi^*(f^i) f_i$ satisfies our requirements. However, in the general case, its lifting would allow "nonsense morphisms", and there would be no analogon to Cor. 1.3.2.

Given a fixed morphism (2.6.1), the formulas (2.6.2), (2.6.3) are applicable also to G-valued superfunctions where G is an arbitrary complete $\mathbb{Z}_2$-lcs. We get a pullback map

$$\phi^* : \mathcal{O}_V^G \to \phi_* \mathcal{O}_U^G \,. \tag{2.6.4}$$

In particular, $\hat{\phi}$ is now simply given as the pullback of the standard coordinate:

$$\hat{\phi} = \phi^*(x_V).$$

Conversely, given an element $\hat{\phi} \in \mathcal{M}^F(U)$, it determines a unique superdomain morphism $\phi : (U, \mathcal{O}_U) \to \mathrm{L}(F)$. Thus, we get a bijection of sets

$$\mathcal{M}^F(U) \to \{\text{morphisms } \ (U, \mathcal{O}_U) \to \mathrm{L}(F)\}, \qquad \phi \mapsto \hat{\phi}. \tag{2.6.5}$$

Now, given (2.6.1) and another morphism of superdomains $\psi : (V, \mathcal{O}_V) \to (W, \mathcal{O}_W)$ then the composite $\psi \circ \phi$ is a morphism, too; we have

$$\widehat{\psi \circ \phi} = \phi^*(\hat{\psi}).$$

Hence the superdomains form a category.

Remark . (2.6.5) shows that for each F, the cofunctor $(U, \mathcal{O}_U) \to \mathcal{M}^F(U)$ is represented by the linear superspace $\mathrm{L}(F)$; the universal element is just the standard coordinate $x \in \mathcal{M}^F(\mathrm{L}(F))$.

Fix $\mathbb{Z}_2$-lcs' E, F. Every even, continuous map $\alpha : E \to F$ yields a morphism of superdomains

$$\mathrm{L}(\alpha) : \mathrm{L}(E) \to \mathrm{L}(F), \quad \widehat{\mathrm{L}(\alpha)} := \alpha \circ x_E \in \mathcal{M}^F(\mathrm{L}(E)).$$

Of course, the underlying map is the restriction of α to $E_{\mathbf{0}}$ while the superfunction pullback is given by

$$(\mathrm{L}(\alpha)^*(u)_e)_{(k|l)} : \prod^k E_{\mathbf{0}} \times \prod^l E_{\mathbf{1}} \xrightarrow{\prod^{k+l} \alpha} \prod^k F_{\mathbf{0}} \times \prod^l F_{\mathbf{1}} \xrightarrow{(u_e)_{(k|l)}} \mathbb{C}$$

for $u \in \mathcal{O}(\alpha(U)), \;\; e \in U, \;\; k, l \geq 0$.

We call any morphism ϕ of superdomains $\mathrm{L}(E) \supseteq U \xrightarrow{\phi} V \to \mathrm{L}(F)$ which is the restriction of some $\mathrm{L}(\alpha)$ a *linear morphism.*

2.7. **Supermanifolds.** In principle, a supermanifold (abbreviated smf) X is a hermitian ringed space which locally looks like a superdomain. However, such a definition would be insufficient because we cannot guarantee a priori that the arising transition map $(U, \mathcal{O}_U) \to (U', \mathcal{O}_{U'})$ between two local models is a morphism of superdomains. Therefore we add to the structure an atlas of local models. That is, we define a *supermanifold* $X = (\mathrm{space}(X), \mathcal{O}_X, (c_i)_{i \in I})$ *(modelled over the complete $\mathbb{Z}_2$-lcs E)* as consisting of the following data:

(1) a Hausdorff space space(X),

(2) a sheaf of hermitian algebras $\mathcal{O} = \mathcal{O}_X$ on it,

(3) a family of *charts*, i. e. of isomorphisms of hermitian ringed spaces

$$c_i : (U_i, \mathcal{O}_X |_{U_i}) \to (c_i(U_i), \mathcal{O}) \tag{2.7.1}$$

where $(U_i)_{i \in I}$ is an open covering of X, and the $(c_i(U_i), \mathcal{O})$ are superdomains modelled over E; in particular, $c_i(U_i) \subseteq E_{\mathbf{0}}$ is open.

Setting $U_{ij} := U_i \cap U_j$, there arise transition morphisms between the local models,

$$g_{ij} := c_i c_j^{-1} : (c_j(U_{ij}), \mathcal{O}) \to (c_i(U_{ij}), \mathcal{O}) \tag{2.7.2}$$

and we require them to be morphisms of superdomains.

Remark . For a formulation without charts cf. [41]. For the purposes of the "synthetic" approach presented here, which explicitly indicates the form of morphisms to be allowed, the point distributions of [41] are not needed at all.

One could also use an approach via Douady's "functored spaces" (cf. [15]) in order to save charts.

The simplest example arises when there is just one chart, i. e. space(X) is effectively an open set $X \subseteq E_{\mathbf{0}}$. We then call X a *superdomain*; for $X = E_{\mathbf{0}}$ we get the linear supermanifold $\mathrm{L}(E)$.

Fix an smf X. In order to define the *sheaf* $\mathcal{O}^F$ *of* F*-valued superfunctions on* X we follow the usual "coordinate philosophy" of differential geometry: Working on an open $U_i \subseteq X$ means actually to work on the superdomain $c_i(U_i) \subseteq \mathrm{L}(E)$; the cocycle (g_{ij}) tells us how to pass from U_i to U_j.

Thus, for open $U \subseteq X$, an element of $\mathcal{O}_X^F(U)$ is a family $(u_i)_{i\in I}$ of elements

$$u_i \in \mathcal{O}^F(c_i(U \cap U_i)) \tag{2.7.3}$$

which satisfies the compatibility condition $g_{ij}^*(u_i) = u_j$ on $c_j(U \cap U_{ij})$.

To make that more explicit, let $x_i \in \mathcal{M}^E(c_i(U_i))$ be the standard coordinate. Then u is given on $U \cap U_i$ by a superfunction (2.7.3) in the sense of 2.4, $u_i = u_i[x_i]$, and compatibility now means

$$u_j[x_j] = u_i[\hat{g}_{ij}[x_j]]$$

on $c_j(U \cap U_{ij})$. Here $\hat{g}_{ij} = g_{ij}^*(x_i) \in \mathcal{M}(c_j(U_{ij}))$.

$\mathcal{O}_X^F$ is a sheaf of hermitian vector spaces.

For fixed j we get an isomorphism of sheaves

$$\mathcal{O}_X^F\,|_{U_i} \to c_j^{-1}(\mathcal{O}_E^F), \qquad u \mapsto c_j^{-1}(u_i).$$

Also, we get an isomorphism

$$\mathcal{O} = \mathcal{O}^{\mathbb{R}}, \qquad v \mapsto \left((c_i^{-1})^*(v)\right),$$

and we can identify both sides.

The product of 2.1 globalizes in an obvious way to a bilinear map

$$\mathcal{O}^F(U) \times \mathcal{O}^G(U) \to \mathcal{O}^{F\otimes G}(U). \tag{2.7.4}$$

Now, given a bilinear map $\alpha : F \times G \to H$ with a third $\mathbb{Z}_2$-lcs H, we can compose (2.7.4) with the map induced by $\alpha : F \otimes G \to H$ to get a bilinear map

$$\mathcal{O}^F(U) \times \mathcal{O}^G(U) \to \mathcal{O}^H(U). \tag{2.7.5}$$

It is natural to denote the image of (u, v) simply by $\alpha(u, v)$.

2.8. **Morphisms of smf's.** A *morphism of smfs* $\mu : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of hermitian ringed spaces which is "compatible with the chart structure". That is, we require that if

$$c_i : (U_i, \mathcal{O}_X\,|_{U_i}) \to (c_i(U_i), \mathcal{O}), \qquad d_j : (V_j, \mathcal{O}_Y\,|_{V_j}) \to (d_j(V_j), \mathcal{O})$$

are charts on X, Y then the arising composite morphism of hermitian ringed spaces

$$(c_i(\mu^{-1}(V_j) \cap U_i), \mathcal{O}) \xrightarrow{c_i^{-1}} X \xrightarrow{\mu} Y \xrightarrow{d_j} (d_j(V_j), \mathcal{O}) \tag{2.8.1}$$

is a morphism of superdomains.

It is easy to check that the supermanifolds form a category; the superdomains now form a full subcategory.

Fix a morphism $\mu : X \to Y$. Globalizing (2.6.4), we define the *pullback of* $\mathcal{O}^G$,

$$\mu^* : \mathcal{O}_Y^G \to \mu_*(\mathcal{O}_X^G)$$

as follows: Let $v \in \mathcal{O}_Y^G(V)$, i. e. $v = (v_j)$ with $v_j = v_j(x_j) \in \mathcal{O}^G(d_j(V \cap V_j))$. We put together the pullbacks of the v_i under the superdomain morphisms (2.8.1): Fixing i, the elements

$$(d_j \mu c_i^{-1})^*(v_i) \in \mathcal{O}^G(c_i(\mu^{-1}(V \cap V_j) \cap U_i))$$

with running j fit together to an element

$$u_i \in \mathcal{O}^G(c_i(\mu^{-1}(V) \cap U_i));$$

and $\mu^*(v) := (u_i) \in \mathcal{O}^G(\mu^{-1}(V))$ is the pullback wanted.

The bijection (2.6.5) globalizes to smfs, yielding the infinite-dimensional version of Cor. 1.3.2:

Theorem 2.8.1. *A morphism of an smf X into a linear smf $\mathrm{L}(E)$ is characterized by the pullback of the standard coordinate x_E. That is, we have a bijection*

$$\mathrm{Mor}(X, \mathrm{L}(E)) \to \mathcal{M}^E(X), \qquad \mu \mapsto \hat{\mu} := \mu^*(x_E).$$

□

This is the superanalogon of the (tautological) non-super fact that a real-analytic function on a manifold M with values in a vector space F is the same as a real-analytic map $M \to F$ (cf. also [41, Thm. 3.4.1]).

2.9. **Supermanifolds and manifolds.** We now turn to the relations of smfs with ordinary real-analytic manifolds modelled over locally convex spaces; since we defined in 2.5 real-analytic maps, this notion makes obvious sense.

Fix an smf X and charts (2.7.1). It follows from 2.5 that the underlying map $\tilde{g}_{ij} : c_j(U_{ij}) \to c_i(U_{ij})$ of each transition morphism (2.7.2) is real-analytic; therefore the maps $\tilde{c}_i : U_i \to E_{\mathbf{0}}$ equip space(X) with the structure of a real-analytic manifold with model space $E_{\mathbf{0}}$ which we call the *underlying manifold* of X and denote by $\tilde{X}$. Assigning also to every smf morphism its underlying map we get a functor

$$\mathbf{Smfs} \to \mathbf{Mfs}, \qquad X \mapsto \tilde{X} \tag{2.9.1}$$

from the category of supermanifolds to the category of manifolds.

We want to construct a right inverse to (2.9.1), i. e. we want to assign to every manifold Y a supermanifold, denoted Y again, the underlying manifold of which is Y again. First we note that given an lcs F, we may view it as $\mathbb{Z}_2$-lcs by $F_{\mathbf{1}} := 0$. Let $\mathrm{An}_F(\cdot)$ be the sheaf of all real-analytic $\mathbb{C}$-valued functions on Y with the obvious hermitian algebra structure. We recall from Thm. 2.5.2.(ii) that we have an isomorphism of hermitian algebra sheaves $\mathrm{An}_F(\cdot) \cong \mathcal{O}(\cdot)$.

Now let be given a real-analytic manifold Y with model space F. We want to assign to Y a supermanifold $Y = (\mathrm{space}(Y), \mathcal{O}_Y, (c_i)_{i \in I})$ with model space F (which we view by $F = F_{\mathbf{0}}$ as $\mathbb{Z}_2$-lcs). Of course, we take the space Y as underlying space, and the sheaf $\mathcal{O}_Y(\cdot) := \mathrm{An}(\cdot, \mathbb{C})$ of real-analytic $\mathbb{C}$-valued functions on Y with the obvious hermitian algebra structure as structure sheaf. In order to get the needed atlas $(c_i)_{i \in I}$, we choose an atlas on the manifold Y in the usual sense, i. e. a collection of open injective maps $c_i' : V_i \to F$ where $Y = \bigcup_{i \in I} V_i$ such that both c_i' and $(c_i')^{-1} : c_i'(V_i) \to V_i$ are real-analytic. Using the usual function pullback we can view the c_i as morphisms of hermitian ringed spaces

$$c_i : (V_i, \mathcal{O}_Y |_{V_i}) = (V_i, \mathrm{An}_{V_i}(\cdot, \mathbb{C})) \to (c_i'(V_i), \mathrm{An}_{c_i'(V_i)}(\cdot, \mathbb{C})) = (c_i'(V_i), \mathcal{O}_F |_{c_i'(V_i)});$$

we claim that these morphisms equip Y with an smf structure. It is sufficient to show that given a real-analytic map $\phi : U \to U'$ with $U, U' \subseteq F$ open, the arising morphism of hermitian ringed spaces

$$\phi : (U, \mathcal{O}_F |_U) \to (U', \mathcal{O}_F |_{U'})$$

is a morphism of superdomains. Indeed, recalling that $\mathcal{M}^F$ is just the sheaf of F-valued analytic maps, we can view ϕ as an element of $\mathcal{M}^F(U)$, and this is the element required in the definition of superdomain morphisms.

One easily shows that this construction is functorial and provides a right inverse

$$\mathbf{Mfs} \to \mathbf{Smfs} \tag{2.9.2}$$

to (2.9.1). Henceforth, we will not make any notational distinction between a manifold Y and the corresponding smf. In particular, any purely even locally convex space E can be viewed as manifold, and hence as smf E.

In fact, the functor (2.9.2) identifies the category of real-analytic manifolds with the full subcategory of all those supermanifolds the model space of which is purely even.

Now let X again be an smf. For each $\mathbb{Z}_2$-lcs F, we get by assigning to each F-valued superfunction its underlying function a sheaf morphism

$$\mathcal{O}^F \to \mathrm{An}(\cdot, F_{\mathbb{C}}), \qquad u \mapsto \tilde{u}; \tag{2.9.3}$$

by Thm. 2.5.2.(ii), it is surjective on every open piece of X which is isomorphic with a superdomain. Also, (2.9.3) restricts to a sheaf morphism $\mathcal{M}^F \to \mathrm{An}(\cdot, F)$.

Now, applying onto X first the functor (2.9.1) and then the functor (2.9.2), we get another smf $\tilde{X}$, and, specializing (2.9.3) to $F := \mathbb{R}$, we get a sheaf morphism $\mathcal{O}_X \to \mathrm{An}(\cdot, \mathbb{C}) = \mathcal{O}_{\tilde{X}}$, and hence a morphism of hermitian ringed spaces

$$(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to (X, \mathcal{O}_X) \tag{2.9.4}$$

the underlying point map of which is the identity. In fact, one shows by looking at the local model that (2.9.4) is an smf morphism

$$\tilde{X} \to X \tag{2.9.5}$$

which we call the *canonical embedding*. It is functorial again: For a given morphism $\mu : X \to Y$ of smfs we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ \uparrow & & \uparrow \\ \tilde{X} & \xrightarrow{\tilde{\mu}} & \tilde{Y}. \end{array}$$

Remarks . (1) Of course, the term "canonical embedding" is slightly abusive as long as the notion of embedding for smfs has not been defined.

(2) The functoriality of (2.9.5) is somewhat deceptive because it does not extend to *families* of morphisms: Given a Z-family of morphisms from X to Y, i. e. a morphism $\mu : Z \times X \to Y$ (cf. below for the definition of the product), there is no natural way to assign to it Z-family of morphisms from $\tilde{X}$ to $\tilde{Y}$, i. e. a morphism $Z \times \tilde{X} \to \tilde{Y}$. All what one has is $\tilde{\mu} : \tilde{Z} \times \tilde{X} \to \tilde{Y}$, which is only a $\tilde{Z}$-family.

2.10. **Cocycle description; products.** Let us describe supermanifolds in terms of cocycles of superfunctions: Assume that we are given the model space E and a manifold $\tilde{X}$ with model space $E_{\mathbf{0}}$ together with charts

$$\tilde{c}_i : U_i \to \tilde{c}_i(U_i) \subseteq E_{\mathbf{0}}. \tag{2.10.1}$$

Let $\tilde{g}_{ij} := \tilde{c}_i \tilde{c}_j^{-1}$; thus $\tilde{g}_{ij} \circ \tilde{g}_{jk} = \tilde{g}_{ik}$ on $\tilde{c}_k(U_i \cap U_j \cap U_k)$.

Now we want to make $\tilde{X}$ the underlying manifold of a supermanifold X which is on each U_i isomorphic to a superdomain in $\mathrm{L}(E)$ such that the maps (2.10.1) are the underlying maps of corresponding charts on X. For this purpose, we have to choose a family $(\hat{g}_{ij})_{i,j \in I}$ of elements $\hat{g}_{ij} \in \mathcal{M}^E(\tilde{c}_j(U_{ij}))$ with underlying function $\tilde{g}_{ij}$ and such that $\hat{g}_{ii} = x_E$,

$$\hat{g}_{ij} \circ \hat{g}_{jk} = \hat{g}_{ik}$$

on $\tilde{c}_k(U_i \cap U_j \cap U_k)$.

For constructing an smf out of these data, we adapt the gluing prescription given in 2.7 to produce a hermitian algebra sheaf $\mathcal{O}_X$: For open $U \subseteq X$, an element of $\mathcal{O}_X(U)$ is a family $(u_i)_{i\in I}$ of elements $u_i \in \mathcal{O}_{\mathrm{L}(E)}(\tilde{c}_i(U \cap U_i))$ which satisfies $u_i \circ \hat{g}_{ij} = u_j$ on $\tilde{c}_j(U \cap U_{ij})$. Moreover, we construct a chart

$$c_i : (U_i, \mathcal{O}_X|_{U_i}) \to (\tilde{c}_i(U_i), \mathcal{O}_{\mathrm{L}(E)}|_{\tilde{c}_i(U_i)})$$

by taking, of course, $\tilde{c}_i$ as underlying map and $u \mapsto (u \circ \hat{g}_{ij})_{j\in I}$ as the superfunction pullback. In particular, we get the local coordinate $x_i := c_i^*(x_E)$, so that

$$x_j = g_{ji} \circ x_i. \tag{2.10.2}$$

It is clear that $X := (\tilde{X}, \mathcal{O}_X, (c_i)_{i\in I})$ is the smf wanted; we call $(\hat{g}_{ij})$ the *defining cocycle* of X.

Remark . Given the data E, $\tilde{X}$, $\tilde{c}_i : U_i \to \tilde{c}_i(U_i) \subseteq E_{\mathbf{0}}$ as above, one obvious lifting $\tilde{g}_{ij} \mapsto \hat{g}_{ij}$ of the transition maps does always exist; it is easily seen to produce the product smf $\tilde{X} \times \mathrm{L}(E_{\mathbf{1}})$ (cf. below for the definition).

We now show that the category of smfs has finite products:

Proposition 2.10.1. *Given smfs X, Y there exists an smf $X \times Y$ together with morphisms $\mathrm{pr}_X : X \times Y \to X$, $\mathrm{pr}_Y : X \times Y \to Y$ such that the following universal property is satisfied:*

Given an smf Z and morphisms $\alpha : Z \to X$, $\beta : Z \to Y$, there exists a unique morphism $\zeta : Z \to X \times Y$ such that $\mathrm{pr}_X \circ\zeta = \alpha$, $\mathrm{pr}_Y \circ\zeta = \beta$.

We call $X \times Y$ the product of X and Y, *and* $\mathrm{pr}_X, \mathrm{pr}_Y$ *the* projections.

Proof. We first treat the case that X, Y are superdomains $U \subseteq \mathrm{L}(E)$, $V \subseteq \mathrm{L}(F)$. Of course, the product sought is just the superdomain $U \times V \subseteq \mathrm{L}(E \oplus F)$ the underlying domain of which is $\mathrm{space}(U) \times \mathrm{space}(V)$; we now define the projection $\mathrm{pr}_U : U \times V \to U$ to be the linear morphism (cf. 2.6) induced by the projection $E \oplus F \to E$. pr_V is given quite analogously.

Now, given $\alpha : Z \to U$, $\beta : Z \to U$, we get elements $\hat{\alpha} \in \mathcal{M}^E(Z) \subseteq \mathcal{M}^{E\oplus F}(Z)$, $\hat{\beta} \in \mathcal{M}^F(Z) \subseteq \mathcal{M}^{E\oplus F}(Z)$, and it is easy to see that the morphism $\zeta : Z \to U \times V$ given by $\hat{\zeta} := \hat{\alpha} + \hat{\beta}$ implements the universal property wanted.

Turning to the case of arbitrary smfs, we use the cocycle construction of above: Let be given smfs X, Y with model spaces E, F, and charts c_i on U_i, d_j on V_j, respectively. We get cocycles

$$\hat{g}_{ik} := \widehat{c_i c_k^{-1}} \in \mathcal{M}^E(\tilde{c}_k(U_{ik})), \quad \hat{h}_{jl} := \widehat{d_j d_l^{-1}} \in \mathcal{M}^F(\tilde{d}_l(V_{jl})).$$

Now $\tilde{X} \times \tilde{Y}$ is a manifold with charts $\tilde{c}_i \times \tilde{d}_j : \mathrm{space}(U_i) \times \mathrm{space}(V_j) \to E_{\mathbf{0}} \oplus F_{\mathbf{0}}$ and transition functions

$$\tilde{c}_i \times \tilde{d}_j(\tilde{c}_k \times \tilde{d}_l)^{-1} = \tilde{g}_{ik} \times \tilde{h}_{jl}.$$

Using the lifting mechanism of above, we can lift $\hat{g}_{ik}$, $\hat{h}_{jl}$ to elements $\hat{g}_{ik}$, $\hat{h}_{jl} \in \mathcal{M}^{E\oplus F}(\tilde{c}_k(U_{ik}) \times \tilde{d}_l(U_{jl}))$, and we take $\hat{g}_{ik}\hat{h}_{jl}$ as the defining cocycle of the supermanifold $X \times Y$ with underlying manifold $\tilde{X} \times \tilde{Y}$. This smf has charts $e_{ij} : (\mathrm{space}(U_i) \times \mathrm{space}(V_j), \mathcal{O}_{X\times Y}) \to c_i(U_i) \times d_j(V_j)$; the compositions

$$\mathrm{space}(U_i) \times \mathrm{space}(V_j) \xrightarrow{\mathrm{pr}_1 \circ e_{ij}} c_i(U_i) \xrightarrow{c_i^{-1}} U_i \xrightarrow{\subseteq} X$$

agree on the overlaps and therefore glue together to an smf morphism $\mathrm{pr}_X : X \times Y \to X$; one defines pr_Y analogously.

Using the superdomain case and some chart juggling, it follows that our requirements are satisfied. □

Remark . The category of smfs has also a terminal object $P = \mathrm{L}(0)$ which is simply a point.

2.11. **Comparison with finite-dimensional Berezin smfs.** Let X be an smf with model space E, and F be any $\mathbb{Z}_2$-lcs. We get an embedding of sheaves

$$\mathcal{O}(\cdot) \otimes F \to \mathcal{O}^F(\cdot), \quad u \otimes f \mapsto (-1)^{|f||u|} f \cdot u \tag{2.11.1}$$

where, of course, on every coordinate patch

$$((f \cdot u)_e)_{(k|l)}(e_1, \dots, e_k, e'_1, \dots, e'_l) = f \cdot (u_e)_{(k|l)}(e_1, \dots, e_k, e'_1, \dots, e'_l)$$

for $e_1, \dots, e_k \in E_{\mathbf{0}}$, $e'_1, \dots, e'_l \in E_{\mathbf{1}}$.

For arbitrary F, (2.11.1) is far away from being isomorphic; in fact, the image consists of all those $u \in \mathcal{O}^F(\cdot)$ for which there exists locally a finite-dimensional $\mathbb{Z}_2$-graded subspace $F' \subseteq F$ such that $u \in \mathcal{O}^{F'}(\cdot)$.

It follows that if F is itself finite-dimensional then (2.11.1) is an isomorphism; this is the reason why the sheaves $\mathcal{O}^F(\cdot)$ become important only in the infinite-dimensional context.

From Thm. 2.8.1 we now get the usual characterization of morphisms by coordinate pullbacks (cf. Thm. 1.3.1).

Corollary 2.11.1. *Let X be an smf, let F be a finite-dimensional $\mathbb{Z}_2$-graded vector space, and let $f_1, \dots, f_k, f'_1, \dots, f'_l \in F^* \subseteq \mathcal{O}(\mathrm{L}(F))$ be a basis of the dual F^*.*

Given elements $u_1, \dots, u_k \in \mathcal{M}(X)$, $v_1, \dots, v_l \in \mathcal{O}(X)_{\mathbf{1},\mathbb{R}}$ there exists a unique smf morphism $\mu : X \to \mathrm{L}(F)$ with the property $\mu^*(f_i) = u_i$, $\mu^*(f'_j) = v_j$ ($i = 1, \dots, k$, $j = 1, \dots, l$). It is given by

$$\hat{\mu} = \sum_{i=1}^{k} f^i \cdot u_i + \sum_{j=1}^{l} (f')^i \cdot v_i \in \mathcal{M}^F(X)$$

where $f^1, \dots, f^k, (f')^1, \dots, (f')^l \in F$ is the left dual basis. □

We now turn to the comparison of our smf category with with the category of finite-dimensional real-analytic Berezin smfs. Adapting the definition given in 1.5 to our real-analytic situation, we define a *(hermitian) real-analytic Berezin smf* X as a hermitian ringed space $(\mathrm{space}(X), \mathcal{O})$, where $\mathrm{space}(X)$ is required to be Hausdorff, and which is locally isomorphic to the model space

$$\mathcal{O}(\cdot) = \mathrm{An}(\cdot, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\xi_1, \dots, \xi_n] \tag{2.11.2}$$

where $\xi_1, \dots, \xi_n$ is a sequence of Grassmann variables. The hermitian structure on (2.11.2) is given by (1.5.1) again.

Note that we have silently dropped the usual requirement of paracompactness since we do not know an infinite-dimensional generalization of it.

Now, given an smf $(\mathrm{space}(X), \mathcal{O}_X, (c_i)_{i \in I})$ in the sense of 2.7 whose model space is finite-dimensional, it follows from the isomorphism (2.4.2) that we only need to forget the charts to get a real-analytic Berezin smf $(\mathrm{space}(X), \mathcal{O}_X)$. Conversely, given such a real-analytic Berezin smf, we can take as atlas e. g. the family of all isomorphisms $c : U \to \mathrm{L}(\mathbb{R}^{m|n})$ where $U \subseteq X$ is open; since, as already observed in Rem. 2.6.1, the morphisms of finitedimensional superdomains "care for themselves", this is OK.

Corollary 2.11.2. *We have an equivalence of categories between*

- *the full subcategory of the category of smfs formed by the smfs with finite-dimensional model space, and*
- *the category of hermitian real-analytic Berezin smfs,*

which acts on objects as

$$(\mathrm{space}(X), \mathcal{O}_X, (c_i)_{i \in I}) \mapsto (\mathrm{space}(X), \mathcal{O}_X).$$

□

Finally, we consider a (fairly simple) functional calculus for scalar superfunctions: Fix an smf X. Every element $f \in \mathcal{M}(X)$ can be interpreted as a morphism $f : X \to \mathrm{L}(\mathbb{R}) = \mathbb{R}$ (on the other hand, any odd superfunction f gives rise to a morphism $\Pi f : X \to \mathrm{L}(\Pi\mathbb{R})$, but we will not use that). Now if $F : U \to \mathbb{R}$ is an analytic function on an open set $U \subseteq \mathbb{R}^n$, and if $f_1, \ldots, f_n \in \mathcal{M}(X)$ then one can make sense of the expression $F(f_1, \ldots, f_n)$ provided that

$$(\tilde{f}_1(x), \ldots, \tilde{f}_n(x)) \in U$$

for all $x \in X$. Indeed, in that case, $F(f_1, \ldots, f_n) \in \mathcal{M}(X)$ is the superfunction corresponding to the composite morphism

$$X \xrightarrow{(f_1,\ldots,f_n)} U \xrightarrow{F} \mathbb{R}.$$

One easily shows:

Corollary 2.11.3. *(i) If* $F = F(z_1, \ldots, z_n) = \sum_{|\alpha|\leq N} c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ *is a polynomial then* $F(f_1, \ldots, f_n) = \sum_{|\alpha|\leq N} c_\alpha f_1^{\alpha_1} \cdots f_n^{\alpha_n}$.

(ii) For $i = 1, \ldots, k$ *let* U_i *be open in* $\mathbb{R}^{n_i}$, *let* V *be open in* $\mathbb{R}^k$. *Let* $F_i \in \mathrm{An}(U_i, \mathbb{R})$, $G \in \mathrm{An}(V, \mathbb{R})$, *suppose that* $(F_1(z_1), \ldots, F_k(z_k)) \in V$ *for all* $(z_1, \ldots, z_k) \in U_1 \times \cdots \times U_k$, *and set*

$$H(z_1, \ldots, z_k) := G(F_1(z_1), \ldots, F_k(z_k))$$

for all $(z_1, \ldots, z_k) \in U_1 \times \cdots \times U_k$. *Let* $f_i = (f_i^1, \ldots, f_i^{n_i}) \in \prod^{n_i} \mathcal{M}(X)$ *for* $i = 1, \ldots, k$, *and suppose that the image of* $f_i : X \to \mathbb{R}^{n_i}$ *lies in* U_i. *Then we have the identity*

$$H(f_1, \ldots, f_k) := G(F_1(f_1), \ldots, F_k(f_k))$$

in $\mathcal{M}(X)$. $\square$

Corollary 2.11.4. *An even scalar superfunction* $f \in \mathcal{O}(X)_{\mathbf{0}}$ *is invertible iff* $\tilde{f}(x) \neq 0$ *for all* $x \in X$.

Proof. The condition is clearly necessary. Set $U := \mathbb{R} \setminus \{0\}, \quad F : U \to U, \quad c \mapsto c^{-1}$. We have $\overline{f} f \in \mathcal{M}(X)$ for $f \in \mathcal{O}(X)_{\mathbf{0}}$, and if f satisfies our condition then we may form $f^{-1} := \overline{f} \cdot F(\overline{f} f)$; using the Cor. above we get $f^{-1} f = 1$. $\square$

Thus, each stalk $\mathcal{O}_x$ of the structure sheaf is a (non-Noetherian) local ring.

Also, $\exp(f)$ is defined for any $f \in \mathcal{M}(X)$, and we have the identity $\exp(f + g) = \exp(f)\exp(g)$.

2.12. **Sub-supermanifolds.** Essentially, we will follow here the line of [25].

Let be given smf's X, Y with model spaces E, F, respectively, and an smf morphism $\phi : X \to Y$. We call ϕ *linearizable* at a point $x \in X$ iff, roughly spoken, it looks at x like a linear morphism (cf. 2.6), i.e. iff there exist neighbourhoods $U \ni x$, $V \ni \tilde{\phi}(x)$, superdomains $U' \subseteq \mathrm{L}(E)$, $V' \subseteq \mathrm{L}(F)$, and isomorphisms $i_U : U \to U'$, $i_V : V \to V'$ such that the composite $\phi' := i_V \phi (i_U)^{-1} : U' \to V' \subseteq \mathrm{L}(F)$ is a linear morphism $\mathrm{L}(\alpha)$. We then call $\alpha : E \to F$ the *model map* of ϕ at x. (We note that the model map is uniquely determined up to automorphisms of E, F since it can be identified with the tangent map $\mathrm{T}_x X \to \mathrm{T}_{\tilde{\phi}(x)} Y$; cf. [42].)

We call an even linear map $F \to E$ of $\mathbb{Z}_2$-lcs a *closed embedding* iff it is injective, its image is closed, and the quotient topology on the image is equal to the topology induced by the embedding into F. We call $F \to E$ a *split embedding* iff it is a closed embedding, and there exists a closed subspace $E' \subseteq E$ such that $E = E' \oplus F$ in the topological sense. We recall that this is equivalent with the existence of a linear continuous projection $E \to F$.

Thus, if F is finite-dimensional and E a Frèchet $\mathbb{Z}_2$-lcs then every injective linear map $F \to E$ is a split embedding.

We call a morphism of smf's $\phi : X \to Y$ a *regular closed embedding* if its underlying point map is injective with closed image, and ϕ is linearizable at every point $x \in X$, with the model map at x being a closed embedding.

One easily shows that if $\phi : X \to Y$ is a regular closed embedding then ϕ is a monomorphism in the sense of category theory, i.e. given two distinct morphisms $Z \rightrightarrows X$ the composites $Z \rightrightarrows X \xrightarrow{\phi} Y$ are still different.

Two regular closed embeddings $\phi : X \to Y$, $\phi' : X' \to Y$ are called *equivalent* iff there exists an isomorphism $\iota : X \to X'$ with $\phi' \circ \iota = \phi$; because of the monomorphic property, ι is uniquely determined.

We call any equivalence class of regular closed embeddings into Y a *sub-supermanifold (sub-smf)* of Y, and we call the equivalence class of a given regular closed embedding $\phi : X \to Y$ the *image* of ϕ. (Sometimes, by abuse of language, one calls X itself a sub-smf, with ϕ being understood.)

Finally, we call a sub-smf $\phi : Y \to X$ a *split sub-smf* if its tangential map is everywhere a split embedding.

For example, the canonical embedding (2.9.5) makes $\tilde{X}$ a split sub-smf of X, with the model map being the embedding $E_{\mathbf{0}} \subseteq E$.

Note that the notion of a regular closed embedding is presumably in general not transitive (although we do not know a counterexample). However, it is easy to see that if $\phi : X \to Y$ is linearizable at x, and $Y \subseteq Z$ is a split sub-smf then the composite $X \xrightarrow{\phi} Y$ is linearizable at x, too. In particular, the notion of a split sub-smf is transitive.

Let be given an smf X and a family of superfunctions $u_i \in \mathcal{O}^{F_i}(X)$, $\; i \in I$, where F_i are $\mathbb{Z}_2$-lcs. We call a sub-smf represented by $\phi : Y \to X$ the *sub-smf cut out by the superfunctions* u_i iff the following holds:

We have $\phi^*(u_i) = 0$; moreover, if $\phi' : Y' \to X$ is any other smf morphism with $(\phi')^*(u_i) = 0$ for all i then there exists a morphism $\iota : Z \to X$ with $\phi \circ \iota = \phi'$.

(Note that by the monomorphic property of ϕ, ι is uniquely determined.)

Of course, for a given family (u_i), such a sub-smf needs not to exist; but if it does, it is uniquely determined. Also, using the universal property with $Y' := P$ being a point we get that if Y exists then $\tilde{\phi} : \mathrm{space}(Y) \to \mathrm{space}(X)$ maps $\mathrm{space}(Y)$ homeomorphically onto

$$\{x \in \mathrm{space}(X) : \quad \widetilde{u_i}(x) = 0 \ \forall i \in I\}.$$

However, note that the superfunction pullback $\mathcal{O}_X \to \mathcal{O}_Y$ need not be locally surjective; it is so only in a finite-dimensional context.

Remark . One can show that appropriate formulations of the inverse and implicite function theorems hold provided that the model spaces of the smf's involved are Banach spaces.

2.13. **Example: the unit sphere of a super Hilbert space.** As a somewhat academic example, let us consider the super variant of the unit sphere of a Hilbert space: As in [39], a *super Hilbert space* is a direct sum of two ordinary Hilbert spaces, $H = H_{\mathbf{0}} \oplus H_{\mathbf{1}}$; the scalar product will be denoted by

$$H \times H \to \mathbb{C}, \qquad (g, h) \mapsto \langle \overline{g} | h \rangle$$

(the unusual notation helps to keep track of the action of the second sign rule).

Let H_r denote then underlying real vector space of H.

Let $x \in \mathcal{M}^{H_r}(\mathrm{L}(H_r))$ denote the standard coordinate. Specializing (2.7.5) to the $\mathbb{R}$-bilinear pairing

$$H_r \times H_r \to \mathbb{C}, \qquad (g, h) \mapsto \langle \overline{g} | h \rangle$$

we get a bilinear pairing

$$\mathcal{O}^{H_r}(\cdot)\times\mathcal{O}^{H_r}(\cdot)\to\mathcal{O}(\cdot). \tag{2.13.1}$$

Applying this to (x,x) yields a superfunction denoted by $\|x\|^2\in\mathcal{M}(\mathrm{L}(H_r))$ (the notation looks abusive, but can be justified, cf. [41, 4.4]). The Taylor series of $\|x\|^2$ at zero is given by $(\|x\|^2)_0 = u_{(2|0)}+u_{(0|2)}$ where

$$u_{(2|0)}:(H_r)_{\mathbf{0}}\times(H_r)_{\mathbf{0}}\to\mathbb{C},\qquad (g,h)\mapsto\langle\overline{g}|h\rangle+\langle\overline{h}|g\rangle=2\cdot\mathrm{Re}\,\langle\overline{g}|h\rangle\,,$$
$$u_{(0|2)}:(H_r)_{\mathbf{1}}\times(H_r)_{\mathbf{1}}\to\mathbb{C},\qquad (g,h)\mapsto\langle\overline{g}|h\rangle-\langle\overline{h}|g\rangle=2\cdot\mathrm{Im}\,\langle\overline{g}|h\rangle\,.$$

Of course, the underlying map is $(H_r)_{\mathbf{0}}\to\mathbb{R},\ g\mapsto\|g\|^2$.

Proposition 2.13.1. *Let H be a super Hilbert space, and suppose that $H_{\mathbf{0}}\neq 0$. There exists a sub-smf $\mathbb{S}\xrightarrow{\epsilon}\mathrm{L}(H_r)$ cut out by the element $\|x\|^2-1\in\mathcal{M}(\mathrm{L}(H_r))$. Moreover, we have an isomorphism*

$$\iota:\mathbb{S}\times\mathbb{R}_+\xrightarrow{\cong}\mathrm{L}(H_r)\setminus 0 \tag{2.13.2}$$

where $\mathbb{R}_+=\{c\in\mathbb{R}:\quad c>0\}$ is viewed by 2.9 as smf. We call $\mathbb{S}$ the super unit sphere *of $\mathrm{L}(H_r)$.*

Proof. Looking at the situation with $Z=P$ being a point, we see that if $\mathbb{S}$ exists its underlying manifold can be identified with the usual unit sphere $\tilde{\mathbb{S}}$ of $(H_r)_{\mathbf{0}}$. In order to use stereographic projection, we fix an element $h\in H_{\mathbf{0}}$, $\|h\|=1$; set

$$E:=\{\xi\in H_r:\quad\mathrm{Re}\,\langle\overline{\xi}|h\rangle=0\}.$$

Note $(H_{\mathbf{1}})_r\subset E$; the real $\mathbb{Z}_2$-graded Hilbert space E will be the model space for $\mathbb{S}$. We get an atlas of the manifold $\tilde{\mathbb{S}}$ consisting of two homeomorphisms

$$\tilde{c}_\pm:\tilde{\mathbb{S}}\setminus\{\pm h\}\to E_{\mathbf{0}},$$

$$\xi\mapsto\pm h+\frac{1}{\pm\,\mathrm{Re}\,\langle\overline{\xi}|h\rangle-1}(\pm h-\xi); \tag{2.13.3}$$

the inverse maps are $\tilde{c}_\pm^{-1}(\eta)=\eta\mapsto\pm h+\frac{2}{1+\|\eta\|^2}(\eta\mp h)$. The transition between the charts is

$$\tilde{g}:=\tilde{c}_+\tilde{c}_-^{-1}:E_{\mathbf{0}}\setminus\{0\}\to E_{\mathbf{0}}\setminus\{0\},\quad \eta\mapsto\eta/\|\eta\|^2\,.$$

In accordance with 2.10, we lift $\tilde{g}$ to the superfunction

$$g[y]:=y/\|y\|^2\in\mathcal{M}^E(\mathrm{L}(E)\setminus\{0\}),$$

where $y\in\mathcal{M}^E(\mathrm{L}(E))$ is the standard coordinate. We get an smf $\mathbb{S}$ with underlying manifold $\tilde{\mathbb{S}}$ and model space E. The $\tilde{c}_\pm$ become the underlying maps of two charts

$$c_\pm:\mathbb{S}\setminus\{\pm h\}\to\mathrm{L}(E),$$

and we get coordinates $y_\pm:=\widehat{c_\pm}=c_\pm{}^*(y)\in\mathcal{M}^E(\mathbb{S}\setminus\{\pm h\})$; from (2.10.2) we get

$$y_-=y_+/\|y_+\|^2\,. \tag{2.13.4}$$

Set

$$e_\pm:=\pm h+\frac{2}{1+\|y_\pm\|^2}(y_\pm\mp h)\in\mathcal{M}^{H_r}(\mathbb{S}\setminus\{\pm h\}).$$

Using (2.13.4), one computes that the restrictions of e_-,e_+ onto $\mathbb{S}\setminus\{h,-h\}$ coincide; hence we can define the smf morphism $\epsilon:\mathbb{S}\to\mathrm{L}(H_r)$ by $\hat{\epsilon}|_{\mathbb{S}\setminus\{\pm h\}}=e_\pm$. Of course, the underlying map $\tilde{\epsilon}$ is the inclusion $\tilde{\mathbb{S}}\subset H_{\mathbf{0}}$. Using again the pairing (2.13.1) one computes

$$\epsilon^*(\|x\|^2-1)=\langle\overline{\hat{\epsilon}}|\hat{\epsilon}\rangle^2-1=0. \tag{2.13.5}$$

Now we can define the morphism (2.13.2) by $\hat{\iota}:=t\hat{\epsilon}$ (if being pedantic, one should write $\mathrm{pr}_1^*(t)\,\mathrm{pr}_2^*(\hat{\epsilon})$ instead) where $t\in\mathcal{M}(\mathbb{R}_+)$ is the standard coordinate.

In order to show that (2.13.2) is an isomorphism, we construct morphisms

$$\kappa_\pm : \mathrm{L}(H_r) \setminus (\pm\mathbb{R}_+ h) \to \mathbb{R}_+ \times (\mathbb{S} \setminus \{\pm h\}), \quad {\kappa_\pm}^*((t, y_\pm)) = \Big(\|x\|\,, \quad \pm h + \frac{1}{\pm\frac{\mathrm{Re}\langle\overline{x}|h\rangle}{\|x\|} - 1}(\pm h - \frac{x}{\|x\|})\Big)$$

where the superfunction $\|x\| := \sqrt{\|x\|^2} \in \mathcal{M}(\mathrm{L}(H_r) \setminus 0)$ is defined by the functional calculus of 2.11. Now κ_+, κ_- coincide on the overlap $\mathrm{L}(H_r) \setminus \mathbb{R}h$: One computes

$$\|{\kappa_+}^*(y_+)\|^2 = \frac{-\mathrm{Re}\,\langle\overline{x}|h\rangle - \|x\|}{\mathrm{Re}\,\langle\overline{x}|h\rangle - \|x\|} \tag{2.13.6}$$

and hence

$${\kappa_+}^*(y_-) = {\kappa_+}^*\left(\frac{1}{\|y_+\|^2}y_+\right) = \frac{1}{\|{\kappa_+}^*(y_+)\|^2}{\kappa_+}^*(y_+) = \frac{\mathrm{Re}\,\langle\overline{x}|h\rangle - \|x\|}{-\mathrm{Re}\,\langle\overline{x}|h\rangle - \|x\|}{\kappa_+}^*(y_+) = {\kappa_-}^*(y_-).$$

Hence κ_+, κ_- glue together to a morphism

$$\kappa : \mathrm{L}(H_r) \setminus \{0\} \to \mathbb{R}_+ \times \mathbb{S}$$

and one shows by a brute force calculation that $\iota\kappa = 1_{\mathrm{L}(H_r)\setminus\{0\}}$, $\kappa\iota = 1_{\mathbb{R}_+\times\mathbb{S}}$. Hence (2.13.2) is indeed an isomorphism, and it follows that ϵ is a regular closed embedding which makes $\mathbb{S}$ a sub-supermanifold of $\mathrm{L}(H)$.

We claim that the composite

$$\omega : \mathrm{L}(H_r) \setminus 0 \xrightarrow{\kappa} \mathbb{R}_+ \times \mathbb{S} \xrightarrow{\mathrm{pr}_2} \mathbb{S} \xrightarrow{\epsilon} \mathrm{L}(H_r)$$

is the "normalization morphism"

$$\hat{\omega} = \hat{\omega}[x] = \frac{1}{\|x\|}x.$$

Indeed,

$$\hat{\omega} = \omega^*(x) = \kappa^*{\mathrm{pr}_2}^*\epsilon(x) = \kappa^*(\pm h + \frac{2}{1 + \|y_\pm\|^2}(y_\pm \mp h)) = \pm h + \frac{2}{1 + \|\kappa^*(y_\pm)\|^2}(\kappa^*(y_\pm) \mp h);$$

investing (2.13.6), the result follows after some more steps.

In view of (2.13.5), it remains to prove: Given some smf morphism $\phi : Z \to \mathrm{L}(H_r)$ with

$$\phi^*(\|x\|^2 - 1) = 0,$$

ϕ factors through $\mathbb{S}$. Indeed, we have $1 = \phi^*(\langle\overline{x}|x\rangle) = \left\langle\overline{\hat{\phi}}|\hat{\phi}\right\rangle = \left\|\hat{\phi}\right\|^2$, i.e. $(\left\|\hat{\phi}\right\| - 1)(\left\|\hat{\phi}\right\| + 1) = 0$; by Cor. 2.11.4, the second factor is invertible, and hence

$$\left\|\hat{\phi}\right\| = 1. \tag{2.13.7}$$

In particular, $\mathrm{Im}(\tilde{\phi}) \subseteq \tilde{\mathbb{S}} \subseteq \mathrm{L}(H_r) \setminus 0$. Now we claim that ϕ coincides with the composite

$$Z \xrightarrow{\phi} \mathrm{L}(H_r) \setminus 0 \xrightarrow{\omega} \mathrm{L}(H_r) \setminus 0.$$

Indeed, using (2.13.7),

$$\widehat{\omega\phi} = \frac{1}{\phi^*(\|x\|)}\phi^*(x) = \frac{1}{\|\phi^*(x)\|}\phi^*(x) = \phi^*(x) = \hat{\phi},$$

and now $\phi = \omega\phi = \epsilon\,\mathrm{pr}_2\,\kappa\phi$ provides the factorization wanted. $\square$

Remark . The whole story becomes much more transparent if one looks at the Z-valued points in the sense of 1.7. Given any smf Z, we get from Thm. 2.8.1 an identification of the set $\mathrm{L}(H_r)(Z)$ of Z-valued points of $\mathrm{L}(H_r)$ with $\mathcal{M}^{H_r}(Z)$, and thus the structure of a (purely even) vector space on $\mathrm{L}(H_r)(Z)$. Of course, thus structure varies functorially with Z. Moreover, (2.13.1) yields a bilinear map $\mathrm{L}(H_r)(Z) \times \mathrm{L}(H_r)(Z) = \mathcal{M}^{H_r}(Z) \times \mathcal{M}^{H_r}(Z) \to \mathcal{M}(Z)$, i. e. a scalar product on $\mathrm{L}(H_r)(Z)$ with values in the algebra $\mathcal{M}(Z)$.

Now the Z-valued points of $\mathbb{S}$ identify with those Z-valued points of $\mathrm{L}(H_r)(Z)$ which have squared length equal to one under this scalar product:

$$\mathbb{S}(Z) = \{\xi \in \mathrm{L}(H_r)(Z) : \quad \langle \overline{\xi} | \xi \rangle - 1 = 0; \}.$$

Now the process of stereographic projection can be applied in the vector space $\mathrm{L}(H_r)(Z)$ for each Z, using the constant morphisms $\pm h : Z \to P \xrightarrow{\pm h} \mathrm{L}(H_r)$ as projection centers and the subspace $\mathrm{L}(E)(Z)$ as screen. One gets maps

$$c_\pm^Z : \{\xi \in \mathbb{S}(Z) : \quad (\pm \mathrm{Re} \langle \overline{\xi}, h \rangle - 1) \text{ is invertible}\} = (\mathbb{S} \setminus \{\pm h\})(Z) \to \mathrm{L}(E)(Z)$$

given by the same formula (2.13.3).

These maps vary functorially with Z again; on the other hand, we noted already in 1.7 that the functor (1.7.1) is faithfully full, that is, "morphisms between representable functors are representable". Thus, the maps $c_\pm^Z$ must be induced by suitable morphisms, and these are indeed just our charts $c_\pm$. The other morphisms $\iota, \epsilon, \kappa, \omega$ can be interpreted similarly.

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